

## 問題 3.1

$f(x)$  が  $x = a$  において微分可能であるとすれば

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \left\{ \frac{f(x) - f(a)}{x - a} (x - a) + f(a) \right\} = f'(a) \cdot 0 + f(a) = f(a)$$

が成り立つから、 $f(x)$  は  $x = a$  において連続である。

## 問題 3.2

$$(1) f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0.$$

$$\begin{aligned} (2) f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2. \end{aligned}$$

$$\begin{aligned} (3) f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h} = \lim_{h \rightarrow 0} \left\{ -\sin\left(x + \frac{h}{2}\right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right\} = -\sin x. \end{aligned}$$

別解 加法定理を利用して次のようにしても良い。

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left( \cos x \cdot h \cdot \frac{\cos h - 1}{h^2} - \sin x \cdot \frac{\sin h}{h} \right) = \cos x \cdot 0 \cdot \left( -\frac{1}{2} \right) - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

$$\begin{aligned} (4) f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} = \lim_{h \rightarrow 0} \left( e^x \cdot \frac{e^h - 1}{h} \right) = e^x. \end{aligned}$$

$$\begin{aligned}
 (5) \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\log(x+h) - \log x}{h} = \lim_{h \rightarrow 0} \frac{\log \frac{x+h}{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\log\left(1 + \frac{h}{x}\right)}{h} = \lim_{h \rightarrow 0} \left\{ \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} \right\} = \frac{1}{x}.
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt[3]{x+h} - \sqrt[3]{x}) \left\{ (\sqrt[3]{x+h})^2 + \sqrt[3]{x+h} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2 \right\}}{h \left\{ (\sqrt[3]{x+h})^2 + \sqrt[3]{x+h} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2 \right\}} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt[3]{x+h})^3 - (\sqrt[3]{x})^3}{h \left\{ (\sqrt[3]{x+h})^2 + \sqrt[3]{x+h} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2 \right\}} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h \left\{ (\sqrt[3]{x+h})^2 + \sqrt[3]{x+h} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2 \right\}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{(\sqrt[3]{x+h})^2 + \sqrt[3]{x+h} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2} = \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3\sqrt[3]{x^2}}.
 \end{aligned}$$

### 問題 3.3

$x \rightarrow +0$  のとき  $\frac{1}{x} \rightarrow \infty$  だから,  $e^{\frac{1}{x}} \rightarrow \infty$ . よって

$$\lim_{x \rightarrow +0} \frac{1}{1 + e^{\frac{1}{x}}} = 0. \quad \dots \quad \textcircled{1}$$

また,  $x \rightarrow -0$  のとき  $\frac{1}{x} \rightarrow -\infty$  だから,  $e^{\frac{1}{x}} \rightarrow e^0 = 1$ . よって

$$\lim_{x \rightarrow -0} \frac{1}{1 + e^{\frac{1}{x}}} = 1. \quad \dots \quad \textcircled{2}$$

(1) 上の ① および ② より,

$$\lim_{x \rightarrow +0} f(x) = \lim_{x \rightarrow +0} \frac{x}{1 + e^{\frac{1}{x}}} = 0, \quad \lim_{x \rightarrow -0} f(x) = \lim_{x \rightarrow -0} \frac{1}{1 + e^{\frac{1}{x}}} = 0.$$

が成り立つので,

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0).$$

従って,  $f(x)$  は  $x = 0$  において連続である.

(2)  $x \neq 0$  のとき

$$\frac{f(x) - f(0)}{x - 0} = \frac{\frac{x}{1 + e^{\frac{1}{x}}} - 0}{x - 0} = \frac{1}{1 + e^{\frac{1}{x}}}$$

だから, ①, ② より

$$\lim_{x \rightarrow +0} \frac{f(x) - f(0)}{x - 0} = 0, \quad \lim_{x \rightarrow -0} \frac{f(x) - f(0)}{x - 0} = 1$$

が成り立つ. よって, 極限值  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$  は存在しないので,  $f(x)$  は  $x = 0$  において微分可能でない.

#### 問題 3.4

(1)  $x \neq 0$  のとき

$$0 \leq \left| \frac{f(x) - f(0)}{x - 0} \right| = \left| x \sin \frac{1}{x} \right| \leq |x| \rightarrow 0 \quad (x \rightarrow 0)$$

が成り立つから, はさみうちの原理により

$$\lim_{x \rightarrow 0} \left| \frac{f(x) - f(0)}{x - 0} \right| = 0 \quad \therefore \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

従って,  $f(x)$  は  $x = 0$  において微分可能である.

(2)  $x \neq 0$  のとき

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \left( -\frac{1}{x^2} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

ここで,  $x \rightarrow 0$  とするとき,  $2x \sin \frac{1}{x} \rightarrow 0$  であるが,  $\cos \frac{1}{x}$  は 0 と 1 の間をいつまでも往復し続けるので, 極限值

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left( 2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

は存在しない. よって,  $f'(x)$  は  $x = 0$  において連続でない.

問題 3.5

$$(1) f(x) = \frac{1}{x} = x^{-1} \text{ だから } f'(x) = -x^{-2} = -\frac{1}{x^2} .$$

$$(2) f(x) = \frac{1}{x^2} = x^{-2} \text{ だから } f'(x) = -2x^{-3} = -\frac{2}{x^3} .$$

$$(3) f(x) = x^2\sqrt{x} = x^{\frac{5}{2}} \text{ だから } f'(x) = \frac{5}{2}x^{\frac{3}{2}} = \frac{5x\sqrt{x}}{2} .$$

$$(4) f(x) = \frac{1}{x\sqrt[3]{x}} = x^{-\frac{4}{3}} \text{ だから } f'(x) = -\frac{4}{3}x^{-\frac{7}{3}} = -\frac{4}{3x^2\sqrt[3]{x}} .$$

問題 3.6

$$(1) y' = 10x^4 - 9x^2 + 10x .$$

$$(2) y' = 2(5x^2 + x - 4) + (2x - 1)(10x + 1) = 30x^2 - 6x - 9 .$$

$$(3) y' = 2x \log x + x^2 \cdot \frac{1}{x} = 2x \log x + x = x(2 \log x + 1) .$$

$$(4) y' = -\frac{(x^3 + 1)'}{(x^3 + 1)^2} = -\frac{3x^2}{(x^3 + 1)^2} .$$

$$(5) y = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \text{ よリ}$$

$$y' = \frac{(\cos x)' \cdot \sin x - \cos x \cdot (\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} .$$

$$(6) y' = \frac{(2x - 1)(x^2 + x + 1) - (x^2 - x + 1)(2x + 1)}{(x^2 + x + 1)^2} = \frac{2(x^2 - 1)}{(x^2 + x + 1)^2} .$$

$$(7) y' = \frac{(\cos x + \sin x)(\sin x + \cos x) - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x)^2 + (\sin x - \cos x)^2}{(\sin x + \cos x)^2} = \frac{2}{1 + 2 \sin x \cos x} = \frac{2}{1 + \sin 2x} .$$

$$(8) y' = \frac{(xe^x)'(1 + \log x) - xe^x(1 + \log x)'}{(1 + \log x)^2} = \frac{(e^x + xe^x)(1 + \log x) - xe^x \cdot \frac{1}{x}}{(1 + \log x)^2}$$

$$= \frac{e^x(1 + x)(1 + \log x) - e^x}{(1 + \log x)^2} = \frac{e^x(x \log x + x + \log x)}{(1 + \log x)^2} .$$

### 問題 3.7

$$(1) y' = 5(x+1)^4 \cdot (x+1)' = 5(x+1)^4 .$$

$$(2) y' = 8(4x+3)^7 \cdot (4x+3)' = 32(4x+3)^7 .$$

$$(3) y = \frac{1}{(3x-1)^4} = (3x-1)^{-4} \text{ だから}$$

$$y' = -4(3x-1)^{-5} \cdot (3x-1)' = -12(3x-1)^{-5} = -\frac{12}{(3x-1)^5} .$$

$$(4) y' = 3(3x^2+4x-1)^2 \cdot (3x^2+4x-1)' = 3(3x^2+4x-1)^2(6x+4) \\ = 6(3x+2)(3x^2+4x-1)^2 .$$

$$(5) y = \sqrt{2x+5} = (2x+5)^{\frac{1}{2}} \text{ だから}$$

$$y' = \frac{1}{2}(2x+5)^{-\frac{1}{2}} \cdot (2x+5)' = \frac{1}{2}(2x+5)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x+5}} .$$

$$(6) y = \frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}} \text{ だから}$$

$$y' = -\frac{1}{2}(4-x)^{-\frac{3}{2}} \cdot (4-x)' = \frac{1}{2}(4-x)^{-\frac{3}{2}} = \frac{1}{2(4-x)^{\frac{3}{2}}} .$$

$$(7) y' = \sin(7x-3) \cdot (7x-3)' = 7\cos(7x-3) .$$

$$(8) y' = e^{6-2x} \cdot (6-2x)' = -2e^{6-2x} .$$

$$(9) y' = \frac{1}{4-3x} \cdot (4-3x)' = \frac{-3}{4-3x} = \frac{3}{3x-4} .$$

$$(10) y' = e^{2x^2+x-8} \cdot (2x^2+x-8)' = (4x+1)e^{2x^2+x-8} .$$

$$(11) y' = 3\sin^2 x \cdot (\sin x)' = 3\sin^2 x \cos x .$$

$$(12) y' = 8\cos^7 x \cdot (\cos x)' = -8\sin x \cos^7 x .$$

$$(13) y' = 5\sin^4 2x \cdot (\sin 2x)' = 5\sin^4 2x \cdot \cos 2x \cdot (2x)' = 10\sin^4 2x \cos 2x .$$

$$(14) y' = \frac{1}{x^2-3x+2} \cdot (x^2-3x+2)' = \frac{2x-3}{x^2-3x+2} .$$

$$(15) y' = \frac{1}{\sin x} \cdot (\sin x)' = \frac{\cos x}{\sin x} = \frac{1}{\tan x} .$$

$$(16) \quad y' = \frac{1}{\cos x} \cdot (\cos x)' = \frac{-\sin x}{\cos x} = -\tan x .$$

### 問題 3.8

$$(1) \quad y' = (\sin^3 x)' \cdot \cos x + \sin^3 x \cdot (\cos x)' = 3 \sin^2 x \cos^2 x - \sin^4 x \\ = 3 \sin^2 x (1 - \sin^2 x) - \sin^4 x = 3 \sin^2 x - 4 \sin^4 x$$

$$(2) \quad y' = (\sin \log x - \cos \log x) + x(\sin \log x - \cos \log x)' \\ = (\sin \log x - \cos \log x) + x \left( \cos \log x \cdot \frac{1}{x} + \sin \log x \cdot \frac{1}{x} \right) \\ = (\sin \log x - \cos \log x) + (\cos \log x + \sin \log x) = 2 \sin \log x$$

$$(3) \quad y = \sqrt{x + \sqrt{x^2 - 1}} = \left( x + (x^2 - 1)^{\frac{1}{2}} \right)^{\frac{1}{2}} \quad \text{だから}$$

$$y' = \frac{1}{2} \left( x + (x^2 - 1)^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \left( x + (x^2 - 1)^{\frac{1}{2}} \right)' \\ = \frac{1}{2} \left( x + (x^2 - 1)^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \left( 1 + \frac{1}{2} (x^2 - 1)^{-\frac{1}{2}} \cdot 2x \right) \\ = \frac{1}{2\sqrt{x + \sqrt{x^2 - 1}}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{2} \sqrt{\frac{x + \sqrt{x^2 - 1}}{x^2 - 1}}$$

$$(4) \quad y = \sqrt[3]{\sin^3 x - 1} = (\sin^3 x - 1)^{\frac{1}{3}} \quad \text{だから}$$

$$y' = \frac{1}{3} (\sin^3 x - 1)^{-\frac{2}{3}} \cdot (\sin^3 x - 1)' \\ = \frac{1}{3} (\sin^3 x - 1)^{-\frac{2}{3}} \cdot 3 \sin^2 x \cos x = \frac{\sin^2 x \cos x}{(\sin^3 x - 1)^{\frac{2}{3}}}$$

$$(5) \quad y' = \frac{1}{x + \sqrt{x^2 + a}} \cdot \left( x + \sqrt{x^2 + a} \right)' = \frac{1}{x + \sqrt{x^2 + a}} \cdot \left( x + (x^2 + a)^{\frac{1}{2}} \right)' \\ = \frac{1}{x + \sqrt{x^2 + a}} \cdot \left( 1 + \frac{1}{2} (x^2 + a)^{-\frac{1}{2}} \cdot 2x \right) \\ = \frac{1}{x + \sqrt{x^2 + a}} \cdot \frac{\sqrt{x^2 + a} + x}{\sqrt{x^2 + a}} = \frac{1}{\sqrt{x^2 + a}}$$

$$(6) \quad y = \frac{e^x \sqrt{x^2 + 1}}{x^2 - 1} = \frac{e^x (x^2 + 1)^{\frac{1}{2}}}{x^2 - 1} \quad \text{だから}$$

$$\begin{aligned} y' &= \frac{\{e^x (x^2 + 1)^{\frac{1}{2}} + e^x \cdot \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x\} (x^2 - 1) - e^x (x^2 + 1)^{\frac{1}{2}} \cdot 2x}{(x^2 - 1)^2} \\ &= \frac{\{e^x (x^2 + 1) + x e^x\} (x^2 - 1) - 2x e^x (x^2 + 1)}{(x^2 - 1)^2 (x^2 + 1)^{\frac{1}{2}}} \\ &= \frac{e^x \{(x^2 + x + 1)(x^2 - 1) - 2x(x^2 + 1)\}}{(x^2 - 1)^2 (x^2 + 1)^{\frac{1}{2}}} = \frac{e^x (x^4 - x^3 - 3x - 1)}{(x^2 - 1)^2 \sqrt{x^2 + 1}} \end{aligned}$$

### 問題 3.9

$$\begin{aligned} (1) \quad \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{4e^x e^{-x}}{4} = \frac{4}{4} = 1. \end{aligned}$$

$$(2) \quad 1 - \tanh^2 x = 1 - \left(\frac{\sinh x}{\cosh x}\right)^2 = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

(3) 右辺を計算する.

$$\begin{aligned} \sinh x \cosh y &= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} = \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4}, \\ \cosh x \sinh y &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4} \end{aligned}$$

であることより

$$\sinh x \cosh y + \cosh x \sinh y = \frac{2e^{x+y} - 2e^{-x-y}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x + y).$$

(4) これも (3) と同様に考えればよい.

$$\begin{aligned} \cosh x \cosh y &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} = \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}}{4}, \\ \sinh x \sinh y &= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4} \end{aligned}$$

であることより

$$\cosh x \cosh y - \sinh x \sinh y = \frac{2e^{x+y} + 2e^{-x-y}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x + y).$$

$$(5) (\cosh x)' = \left( \frac{e^x + e^{-x}}{2} \right)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$(6) (\sinh x)' = \left( \frac{e^x - e^{-x}}{2} \right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$(7) (\tanh x)' = \left( \frac{\sinh x}{\cosh x} \right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

補充問題 以下の関数を微分せよ。(対数微分法)

$$(1) y = x^{x^2} \qquad (2) y = x^{\sin x} \qquad (3) y = \frac{(x+3)^2}{(x+1)^2(x-2)^4}$$

解答 (1)  $y = x^{x^2}$  の両辺の対数をとれば  $\log y = \log x^{x^2} = x^2 \log x$  となるので

$$\frac{y'}{y} = 2x \log x + x^2 \cdot \frac{1}{x} = x(2 \log x + 1) \quad \therefore y' = x^{x^2+1}(2 \log x + 1).$$

(2)  $y = x^{\sin x}$  の両辺の対数をとれば  $\log y = \log x^{\sin x} = \sin x \log x$  となるので

$$\frac{y'}{y} = \cos x \log x + \frac{\sin x}{x} \quad \therefore y' = x^{\sin x} \left( \cos x \log x + \frac{\sin x}{x} \right).$$

(3)  $y = \frac{(x+3)^2}{(x+1)^2(x-2)^4}$  の両辺の対数をとれば

$$\log y = \log \frac{(x+3)^2}{(x+1)^2(x-2)^4} = 2 \log(x+3) - 2 \log(x+1) - 4 \log(x-2)$$

となるので

$$\begin{aligned} \frac{y'}{y} &= \frac{2}{x+3} - \frac{2}{x+1} - \frac{4}{x-2} \\ &= \frac{2(x+1)(x-2) - 2(x+3)(x-2) - 4(x+3)(x+1)}{(x+3)(x+1)(x-2)} \\ &= \frac{-4x^2 - 20x - 4}{(x+3)(x+1)(x-2)} = -\frac{4(x^2 + 5x + 1)}{(x+3)(x+1)(x-2)}. \end{aligned}$$

従って

$$y' = -\frac{4(x^2 + 5x + 1)}{(x+3)(x+1)(x-2)} \cdot \frac{(x+3)^2}{(x+1)^2(x-2)^4} = -\frac{4(x+3)(x^2 + 5x + 1)}{(x+1)^3(x-2)^5}.$$