『微分積分』 第 7 章 演習問題の解答

問題 7.1

(1) $x = 2\sqrt{2}\sin\theta$ とおくと, $\theta : 0 \to \dfrac{\pi}{4}$ のとき $x : 0 \to 2$ であり,

$$\frac{dx}{d\theta} = 2\sqrt{2}\cos\theta \qquad \therefore \quad dx = 2\sqrt{2}\cos\theta\, d\theta\ .$$

従って,

$$\begin{aligned}\int_0^2 \sqrt{8-x^2}\, dx &= \int_0^{\frac{\pi}{4}} \sqrt{8-8\sin^2\theta}\cdot 2\sqrt{2}\cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} 8\cos^2\theta\, d\theta \\ &= 4\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)\, d\theta = 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= 4\left(\frac{\pi}{4}+\frac{1}{2}\right) = \pi + 2\ .\end{aligned}$$

(2) $x = 2\sin\theta$ とおくと, $\theta : 0 \to \dfrac{\pi}{6}$ のとき $x \to 0 \to 1$ であり,

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \therefore \quad dx = 2\cos\theta\, d\theta\ .$$

従って,

$$\begin{aligned}\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx &= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{6}} 4\sin^2\theta\, d\theta = 2\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\, d\theta \\ &= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}\ .\end{aligned}$$

(3) $x = \sin\theta$ とおくと, $\theta : 0 \longrightarrow \dfrac{\pi}{2}$ のとき $x : 0 \to 1$ であり,

$$\frac{dx}{d\theta} = \cos\theta \qquad \therefore \quad dx = \cos t\, dt\ .$$

従って,

$$\begin{aligned}\int_0^1 x^2\sqrt{1-x^2}\, dx &= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta\ \cos^2\theta\, d\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\, d\theta \\ &= \frac{1}{4}\left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{16}\ .\end{aligned}$$

(4) $x = \tan\theta$ とおけば, $\theta\ :\ -\dfrac{\pi}{4} \to \dfrac{\pi}{4}$ のとき $x\ :\ -1 \longrightarrow 1$ であり,

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta} \qquad \therefore \quad dx = \frac{1}{\cos^2\theta}\,d\theta\ .$$

そして

$$\int_{-1}^{1}\frac{dx}{(1+x^2)^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{(1+\tan^2\theta)^2}\cdot\frac{1}{\cos^2\theta}\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos^4 t}{\cos^2\theta}\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2\theta\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1+\cos 2\theta}{2}\,d\theta = \left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi+2}{4}\ .$$

(5) $t = \cos x$ とおけば, $x\ :\ 0 \to \dfrac{\pi}{2}$ のとき $t\ :\ 1 \to 0$ であり,

$$\frac{dt}{dx} = -\sin x \qquad \therefore \quad dx = -\frac{1}{\sin x}dt\ .$$

従って,

$$\int_{0}^{\frac{\pi}{2}}\frac{\sin^3 x}{1+\cos x}\,dx = \int_{1}^{0}\frac{\sin^3 x}{1+\cos x}\cdot\left(-\frac{1}{\sin x}\right)dt = \int_{0}^{1}\frac{\sin^2 x}{1+\cos x}\,dt$$

$$= \int_{0}^{1}\frac{1-t^2}{1+t}\,dt = \int_{0}^{1}(1-t)\,dt = \left[t-\frac{t^2}{2}\right]_{0}^{1}$$

$$= 1-\frac{1}{2} = \frac{1}{2}\ .$$

(6) $t = e^x$ とおけば, $x\ :\ 0 \to \log 3$ のとき $t\ :\ 1 \to 3$ であり,

$$\frac{dt}{dx} = e^x \qquad \therefore \quad dx = \frac{1}{e^x}\,dt\ .$$

従って,

$$\int_{0}^{\log 3}\frac{e^x}{e^x+e^{-x}}\,dx = \int_{1}^{3}\frac{e^x}{e^x+e^{-x}}\cdot\frac{1}{e^x}\,dt = \int_{1}^{3}\frac{e^x}{e^{2x}+1}\,dt$$

$$= \int_{1}^{3}\frac{t}{t^2+1}\,dt = \left[\frac{1}{2}\log(t^2+1)\right]_{1}^{3}$$

$$= \frac{1}{2}(\log 10 - \log 2) = \frac{1}{2}\log 5\ .$$

(7) $t = e^x$ とおけば, $x\ :\ 0 \to \log 2$ のとき $t\ :\ 1 \to 2$ であり,

$$\frac{dt}{dx} = e^x \qquad \therefore \quad dx = \frac{1}{e^x}\,dt\ .$$

従って,

$$\int_0^{\log 2} \frac{dx}{e^x - 3 - 4e^{-x}} = \int_1^2 \frac{1}{e^x - 3 - 4e^{-x}} \cdot \frac{1}{e^x}\, dt = \int_1^2 \frac{1}{e^{2x} - 3e^x - 4}\, dt$$

$$= \int_1^2 \frac{1}{t^2 - 3t - 4}\, dt = \int_1^2 \frac{1}{(\,t+1\,)(\,t-4\,)}\, dt$$

$$= \frac{1}{5} \int_1^2 \left(\frac{1}{t-4} - \frac{1}{t+1} \right) dt$$

$$= \frac{1}{5} \Big[\log|\,t-4\,| - \log|\,t+1\,| \Big]_1^2 = \frac{1}{5} \left[\log \left| \frac{t-4}{t+1} \right| \right]_1^2$$

$$= \frac{1}{5} \left(\log\frac{2}{3} - \log\frac{3}{2} \right) = \frac{2}{5} \log\frac{2}{3} \,.$$

(8) $t = \sqrt{x+1}$ とおけば, $x : 0 \to 2$ のとき $t : 1 \to \sqrt{3}$ であり,

$$\frac{dt}{dx} = \frac{1}{2\sqrt{x+1}} \qquad \therefore \quad dx = 2\sqrt{x+1}\, dt = 2t\, dt \,.$$

また, $x = t^2 - 1$ だから,

$$\int_0^2 \frac{2x+1}{\sqrt{x+1}}\, dx = \int_1^{\sqrt{3}} \frac{2(\,t^2-1\,)+1}{t}\, 2t\, dt = \int_1^{\sqrt{3}} (\,4t^2 - 2\,)\, dt = \left[\frac{4}{3}t^3 - 2t \right]_1^{\sqrt{3}}$$

$$= \left(\frac{4}{3} \cdot 3\sqrt{3} - 2\sqrt{3} \right) - \left(\frac{4}{3} - 2 \right) = 2\sqrt{3} + \frac{2}{3} \,.$$

(9) $t = \sin x$ と置換したくなるが, それだとうまく行かない.

$t = \cos x$ とおくと $x : \dfrac{\pi}{6} \to \dfrac{\pi}{3}$ のとき $t : \dfrac{\sqrt{3}}{2} \to \dfrac{1}{2}$ であり, $dx = -\dfrac{1}{\sin x}\, dt$ だから

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x} = \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{\sin x} \left(-\frac{1}{\sin x} \right) dt = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sin^2 x}\, dt = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2}\, dt$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) dx = \frac{1}{2} \Big[\log|\,1+t\,| - \log|\,1-t\,| \Big]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left[\frac{1}{2} \log \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2} \log \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} - \frac{1}{2} \log \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{1}{2} \log \frac{2+\sqrt{3}}{2-\sqrt{3}} - \frac{1}{2} \log 3 = \log(\,2+\sqrt{3}\,) - \log\sqrt{3}$$

$$= \log \frac{2+\sqrt{3}}{\sqrt{3}} = \log \frac{3+2\sqrt{3}}{3} \,.$$

(10) $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$ とする.

ここで $t = \dfrac{\pi}{2} - x$ とおけば, $x : 0 \to \dfrac{\pi}{2}$ のとき $t : \dfrac{\pi}{2} \to 0$ であり, $dx = -dt$. 従って,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx = \int_{\frac{\pi}{2}}^0 \frac{\sin t}{\cos t + \sin t} \cdot (-1)\, dt \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin t}{\cos t + \sin t}\, dt = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx . \end{aligned}$$

よって

$$\begin{aligned} 2I &= I + I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\sin x + \cos x}\, dx = \int_0^{\frac{\pi}{2}} dx = \Big[\, x \,\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \end{aligned}$$

を得るので $I = \dfrac{\pi}{4}$.

問題 7.2

(1) $t = \cos x$ とおけば $\dfrac{dt}{dx} = -\sin x$ だから $dx = -\dfrac{dt}{\sin x}$ である. よって

$$\begin{aligned} \int \frac{\sin x}{\sqrt{3 - \cos^2 x}}\, dx &= \int \frac{\sin x}{\sqrt{3 - \cos^2 x}} \cdot (-1) \cdot \frac{dt}{\sin x} = -\int \frac{dt}{\sqrt{3 - t^2}} \\ &= -\sin^{-1} \frac{t}{\sqrt{3}} = -\sin^{-1} \frac{\cos x}{\sqrt{3}} . \end{aligned}$$

(2) $t = \sqrt{e^x - 1}$ とおけば

$$\frac{dt}{dx} = \frac{1}{2}(\, e^x - 1\,)^{-\frac{1}{2}} \cdot e^x = \frac{e^x}{2\sqrt{e^x - 1}} \qquad \therefore \quad dx = \frac{2\sqrt{e^x - 1}}{e^x} dt .$$

よって

$$\begin{aligned} \int \frac{e^x\sqrt{e^x - 1}}{e^x + 3}\, dx &= \int \frac{e^x\sqrt{e^x - 1}}{e^x + 3} \cdot \frac{2\sqrt{e^x - 1}}{e^x}\, dt = \int \frac{2(\, e^x - 1\,)}{e^x + 3}\, dt \\ &= \int \frac{2t^2}{t^2 + 4}\, dt = \int \left(2 - \frac{8}{t^2 + 4} \right) dx = 2t - 8 \cdot \frac{1}{2} \tan^{-1} \frac{t}{2} \\ &= 2\sqrt{e^x - 1} - 4\tan^{-1} \frac{\sqrt{e^x - 1}}{2} . \end{aligned}$$

問題 7.3

(1) $t=\pi-x$ とおけば $dx=-dt$ であり，$x:0\to\pi$ のとき $t:\pi\to 0$ だから

$$\begin{aligned} I &= \int_\pi^0 \frac{(\pi-t)\sin t}{9-\cos^2 t}\cdot(-1)\,dt = \pi\int_0^\pi \frac{\sin t}{9-\cos^2 t}\,dt-\int_0^\pi \frac{t\sin t}{9-\cos^2 t}\,dt \\ &= \pi\int_0^\pi \frac{\sin t}{9-\cos^2 t}\,dt - I\ . \end{aligned}$$

これより

$$I=\frac{\pi}{2}\int_0^\pi \frac{\sin t}{9-\cos^2 t}\,dt=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{9-\cos^2 x}\,dx$$

を得る.

(2) $t=\cos x$ とおけば $dx=-\dfrac{1}{\sin x}\,dt$ であり，$x:0\to\pi$ のとき $t:1\to -1$ なので

$$\begin{aligned} I &= \frac{\pi}{2}\int_1^{-1}\frac{\sin x}{9-\cos^2 x}\cdot\left(-\frac{1}{\sin x}\right)dt = \frac{\pi}{2}\int_{-1}^1\frac{1}{9-\cos^2 x}\,dt \\ &= \frac{\pi}{2}\int_{-1}^1\frac{1}{9-t^2}\,dt = \frac{\pi}{2}\left[\ \frac{1}{6}\log\left|\frac{3+t}{3-t}\right|\ \right]_{-1}^1 = \frac{\pi}{6}\log 2\ . \end{aligned}$$

問題 7.4

(1) $$\int_0^{\frac{1}{\sqrt{2}}}\cos^{-1}x\,dx=\left[\ x\cos^{-1}x\ \right]_0^{\frac{1}{\sqrt{2}}}-\int_0^{\frac{1}{\sqrt{2}}}x\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{1}{\sqrt{2}}\cos^{-1}\frac{1}{\sqrt{2}}-\left[\ \sqrt{1-x^2}\ \right]_0^{\frac{1}{\sqrt{2}}}=\frac{\pi}{4\sqrt{2}}-\left(\frac{1}{\sqrt{2}}-1\right)=\frac{8-4\sqrt{2}+\sqrt{2}\,\pi}{8}\ .$$

(2) $$\int_0^{\sqrt{3}}\tan^{-1}x\,dx=\left[\ x\tan^{-1}x\ \right]_0^{\sqrt{3}}-\int_0^{\sqrt{3}}x\cdot\frac{1}{x^2+1}\,dx$$

$$=\sqrt{3}\tan^{-1}\sqrt{3}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{(x^2+1)'}{x^2+1}\,dx=\frac{\sqrt{3}\,\pi}{3}-\frac{1}{2}\left[\ \log(x^2+1)\ \right]_0^{\sqrt{3}}$$

$$=\frac{\sqrt{3}\,\pi}{3}-\frac{1}{2}\log 4=\frac{\sqrt{3}\,\pi}{3}-\log 2\ .$$

(3) $$\int_0^1\log(x^2+1)\,dx=\left[\ x\log(x^2+1)\ \right]_0^1-\int_0^1 x\cdot\frac{2x}{x^2+1}\,dx$$

$$=\log 2-\int_0^1\left(2-\frac{2}{x^2+1}\right)dx=\log 2-\left[\ 2x-2\tan^{-1}x\ \right]_0^1$$

$$=\log 2-2+2\tan^{-1}1=\log 2-2+\frac{\pi}{2}\ .$$

問題 7.5

(1) $\displaystyle \int x^2 \sin x\, dx = x^2 \cdot (-\cos x) - \int 2x \cdot (-\cos x)\, dx = -x^2 \cos x + 2\int x \cos x\, dx$

$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\, dx\right) = (2 - x^2)\cos x + 2x \sin x\ .$$

(2) $\displaystyle \int x(\log x)^2\, dx = \frac{1}{2}x^2(\log x)^2 - \int \frac{1}{2}x^2 \cdot \log x \frac{1}{x}\, dx$

$$= \frac{1}{2}x^2(\log x)^2 - \int x \log x\, dx = \frac{1}{2}x^2(\log x)^2 - \left(\frac{1}{2}x^2 \log x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx\right)$$

$$= \frac{1}{2}x^2(\log x)^2 - \frac{1}{2}x^2 \log x + \frac{1}{4}x^2 = \frac{1}{2}x^2 \left\{(\log x)^2 - \log x + \frac{1}{2}\right\}\ .$$

(3) これは少し工夫が必要.

$$\begin{aligned} \int e^x \sin x\, dx &= e^x \sin x - \int e^x \cos x\, dx = e^x \sin x - \left(e^x \cos x - \int e^x \cdot (-\sin x)\, dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \end{aligned}$$

であることより

$$\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x)\ .$$

(4)
$$\begin{aligned} \int \sqrt{x^2 + A}\, dx &= x\sqrt{x^2 + A} - \int x \cdot \frac{1}{2}(x^2 + A)^{-\frac{1}{2}} \cdot 2x\, dx \\ &= x\sqrt{x^2 + A} - \int \frac{x^2}{\sqrt{x^2 + A}}\, dx \\ &= x\sqrt{x^2 + A} - \int \frac{(x^2 + A) - A}{\sqrt{x^2 + A}}\, dx \\ &= x\sqrt{x^2 + A} + A\int \frac{dx}{\sqrt{x^2 + A}} - \int \sqrt{x^2 + A}\, dx \\ &= x\sqrt{x^2 + A} + A \log\left| x + \sqrt{x^2 + A} \right| - \int \sqrt{x^2 + A}\, dx \end{aligned}$$

であることより

$$\int \sqrt{x^2 + A}\, dx = \frac{1}{2}\left(x\sqrt{x^2 + A} + A \log\left| x + \sqrt{x^2 + A} \right|\right)\ .$$

問題 7.6

(1) $t=\sqrt{x}$ とおけば $\dfrac{dt}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}$ だから $dx=2\sqrt{x}\,dt=2t\,dt$.

そして $x:0\to 1$ のとき $t:0\to 1$ であるから

$$\begin{aligned}\int_0^1 e^{\sqrt{x}}\,dx &= \int_0^1 e^t\cdot 2t\,dt=2\int_0^1 te^t\,dt=2\Big[\,te^t\,\Big]_0^1-2\int_0^1 e^t\,dt=2e-2\Big[\,e^t\,\Big]_0^1\\ &= 2e-2(\,e-1\,)=2\ .\end{aligned}$$

(2) これもやはり $t=\sqrt{x}$ とおけば $\dfrac{dt}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}$ であり, $dx=2\sqrt{x}\,dt=2t\,dt$.

そして $x:0\to 1$ のとき $t:0\to 1$ であるから

$$\begin{aligned}\int_0^1 \log\big(\,1+\sqrt{x}\,\big)\,dx &= \int_0^1 \log(\,1+t\,)\,2t\,dt \;=\; \Big[\,t^2\,\log(\,1+t\,)\Big]_0^1-\int_0^1\frac{t^2}{1+t}\,dt\\ &= \log 2-\int_0^1\frac{(\,t+1\,)(\,t-1\,)+1}{1+t}\,dt\\ &= \log 2-\int_0^1\left(t-1+\frac{1}{1+t}\right)dt\\ &= \log 2-\left[\,\frac{1}{2}t^2-t+\log|\,1+t\,|\,\right]_0^1\\ &= \log 2-\left(\,\frac{1}{2}-1+\log 2\right)\;=\;\frac{1}{2}\ .\end{aligned}$$