『微分積分』 第 8 章 演習問題の解答

問題 8.1

(1) $\dfrac{x-6}{x^2+3x-4} = \dfrac{x-6}{(x+4)(x-1)} = \dfrac{a}{x+4} + \dfrac{b}{x-1}$ とおけば

$$\frac{x-6}{(x+4)(x-1)} = \frac{(a+b)x + (-a+4b)}{(x+4)(x-1)}$$

であるから

$$a+b=1 \quad , \quad -a+4b=-6 \qquad \therefore \quad a=2 \ , \ b=-1 \, .$$

従って

$$\int \frac{x-6}{x^2+3x-4}\,dx = \int \left(\frac{2}{x+4} - \frac{1}{x-1} \right) dx$$

$$= 2\log|x+4| - \log|x-1| = \log \frac{(x+4)^2}{|x-1|} \, .$$

(2) $\dfrac{2x^2-5x-9}{x^3+2x^2-x-2} = \dfrac{2x^2-5x-9}{(x-1)(x+1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+1} + \dfrac{c}{x+2}$ とおけば

$$\frac{2x^2-5x-9}{(x-1)(x+1)(x+2)} = \frac{(a+b+c)x^2 + (3a+b)x + (2a-2b-c)}{(x-1)(x+1)(x+2)}$$

であるから

$$a+b+c=2 \quad , \quad 3a+b=-5 \quad , \quad 2a-2b-c=-9 \, .$$

$$\therefore \quad a=-2 \ , \ b=1 \ , \ c=3 \, .$$

従って

$$\int \frac{2x^2-5x-9}{x^3+2x^2-x-2}\,dx = -2\int \frac{dx}{x-1} + \int \frac{dx}{x+1} + 3\int \frac{dx}{x+2}$$

$$= -2\log|x-1| + \log|x+1| + 3\log|x+2|$$

$$= \log \left| \frac{(x+1)(x+2)^3}{(x-1)^2} \right| \, .$$

(3) $3x^4+10x^3+5x^2-17x-9$ を x^2+4x+5 で割った商は $3x^2-2x-2$, 余りは $x+1$ であるから

$$3x^4+10x^3+5x^2-17x-9 = (x^2+4x+5)(3x^2-2x-2) + x + 1 \, .$$

よって

$$
\begin{aligned}
&\int \frac{3x^4+10x^3+5x^2-17x-9}{x^2+4x+5}\,dx \\
=&\int \frac{(\,x^2+4x+5\,)(\,3x^2-2x-2\,)+x+1}{x^2+4x+5}\,dx \\
=&\int \left(3x^2-2x-2+\frac{x+1}{x^2+4x+5}\right)dx \\
=&\int (\,3x^2-2x-2\,)\,dx+\frac{1}{2}\int \frac{2x+2}{x^2+4x+5}\,dx \\
=&\int (\,3x^2-2x-2\,)\,dx+\frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx-\int \frac{dx}{x^2+4x+5} \\
=&\int (3x^2-2x-2)\,dx+\frac{1}{2}\int \frac{(\,x^2+4x+5\,)'}{x^2+4x+5}\,dx-\int \frac{dx}{(x+2)^2+1} \\
=&\ x^3-x^2-2x+\frac{1}{2}\log(\,x^2+4x+5\,)-\tan^{-1}(\,x+2\,)\,.
\end{aligned}
$$

(4) $\dfrac{4x^2+5}{x^3-x^2+2x-2}=\dfrac{4x^2+5}{(\,x-1\,)(\,x^2+2\,)}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+2}$ とおけば

$$
\frac{4x^2+5}{(\,x-1\,)(\,x^2+2\,)}=\frac{(\,a+b\,)x^2+(\,-b+c\,)x+(\,2a-c\,)}{(\,x-1\,)(\,x^2+2\,)}
$$

であるから

$$
a+b=4 \quad , \quad -b+c=0 \quad , \quad 2a-c=5
$$

$$
\therefore \quad a=3\ ,\ b=1\ ,\ c=1\,.
$$

従って

$$
\begin{aligned}
\int \frac{4x^2+5}{x^3-x^2+2x-2}\,dx &= \int \left(\frac{3}{x-1}+\frac{x+1}{x^2+2}\right)dx \\
&= \int \frac{3}{x-1}\,dx+\frac{1}{2}\int \frac{2x+2}{x^2+2}\,dx \\
&= \int \frac{3}{x-1}\,dx+\frac{1}{2}\int \frac{2x}{x^2+2}\,dx+\int \frac{1}{x^2+2}\,dx \\
&= 3\log|\,x-1\,|+\frac{1}{2}\log(\,x^2+2\,)+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\,.
\end{aligned}
$$

(5) $\dfrac{1}{x^3-8}=\dfrac{1}{(\,x-2\,)(\,x^2+2x+4\,)}=\dfrac{a}{x-2}+\dfrac{bx+c}{x^2+2x+4}$ とおけば

$$\frac{1}{(\,x-2\,)(\,x^2+2x+4\,)}=\frac{(\,a+b\,)x^2+(\,2a-2b+c\,)x+(\,4a-2c\,)}{(\,x-2\,)(\,x^2+2x+4\,)}$$

であるから

$$a+b=0 \quad , \quad 2a-2b+c=0 \quad , \quad 4a-2c=1\,.$$

$$\therefore \quad a=\frac{1}{12} \;\;,\;\; b=-\frac{1}{12} \;\;,\;\; c=-\frac{1}{3}\,.$$

従って

$$\begin{aligned}
\int\frac{dx}{x^3-8} &= \frac{1}{12}\int\left(\frac{dx}{x-2}-\frac{x+4}{x^2+2x+4}\right)dx \\
&= \frac{1}{12}\left(\int\frac{dx}{x-2}-\frac{1}{2}\int\frac{2x+2}{x^2+2x+4}\,dx-3\int\frac{dx}{x^2+2x+4}\right) \\
&= \frac{1}{12}\left(\int\frac{dx}{x-2}-\frac{1}{2}\int\frac{(x^2+2x+4)'}{x^2+2x+4}\,dx-3\int\frac{dx}{(x+1)^2+3}\right) \\
&= \frac{1}{12}\left(\log|\,x-2\,|-\frac{1}{2}\log(x^2+2x+4)-3\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}}\right) \\
&= \frac{1}{12}\log|\,x-2\,|-\frac{1}{24}\log(x^2+2x+4)-\frac{1}{4\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} \\
&= \frac{1}{24}\log\frac{(x-2)^2}{x^2+2x+4}-\frac{1}{4\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}}\,.
\end{aligned}$$

(6) $\dfrac{x^3+1}{x(\,x-1\,)^3}=\dfrac{x^3+1}{x(\,x-1\,)^3}=\dfrac{a}{x}+\dfrac{b}{x-1}+\dfrac{c}{(\,x-1\,)^2}+\dfrac{d}{(\,x-1\,)^3}$ とおけば

$$\begin{aligned}
\frac{x^3+1}{x(\,x-1\,)^3} &= \frac{a(\,x-1\,)^3+bx(\,x-1\,)^2+cx(\,x-1\,)+dx}{x(\,x-1\,)^3} \\
&= \frac{(\,a+b\,)x^3+(\,-3a-2b+c\,)x^2+(\,3a+b-c+d\,)x-a}{x(\,x-1\,)^3}
\end{aligned}$$

であるから

$$a+b=1 \quad , \quad -3a-2b+c=0 \quad , \quad 3a+b-c+d=0 \quad , \quad -a=1\,.$$

$$\therefore \quad a=-1 \;\;,\;\; b=2 \;\;,\;\; c=1 \;\;,\;\; d=2\,.$$

従って

$$\begin{aligned}\int \frac{x^3+1}{x(x-1)^3}\,dx &= \int\left(-\frac{1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}\right)dx\\ &= -\log|x|+2\log|x-1|-\frac{1}{x-1}-\frac{1}{(x-1)^2}\\ &= \log\frac{(x-1)^2}{|x|}-\frac{x}{(x-1)^2}\,.\end{aligned}$$

(7) 分母を因数分解すれば

$$x^4+1=(x^4+2x^2+1)-2x^2=(x^2+1)^2-2x^2=\left(x^2+\sqrt{2}\,x+1\right)\left(x^2-\sqrt{2}\,x+1\right)$$

となる．そこで

$$\begin{aligned}\frac{1}{x^4+1} &= \frac{1}{\left(x^2+\sqrt{2}\,x+1\right)\left(x^2-\sqrt{2}\,x+1\right)}\\ &= \frac{ax+b}{x^2+\sqrt{2}\,x+1}+\frac{cx+d}{x^2-\sqrt{2}\,x+1}\end{aligned}$$

とおいて, 最後の式を通分すれば

$$\frac{(a+c)x^3+\left(-\sqrt{2}\,a+b+\sqrt{2}\,c+d\right)x^2+\left(a-\sqrt{2}\,b+c+\sqrt{2}\,d\right)x+(b+d)}{\left(x^2+\sqrt{2}\,x+1\right)\left(x^2-\sqrt{2}\,x+1\right)}$$

となるから, 分子の係数を比較することにより

$$a+c=0\,,\quad -\sqrt{2}\,a+b+\sqrt{2}\,c+d=0\,,\quad a-\sqrt{2}\,b+c+\sqrt{2}\,d=0\,,\quad b+d=1\,.$$

$$\therefore\quad a=\frac{1}{2\sqrt{2}}\,,\quad b=\frac{1}{2}\,,\quad c=-\frac{1}{2\sqrt{2}}\,,\quad d=\frac{1}{2}\,.$$

従って

$$\int\frac{dx}{x^4+1} = \frac{1}{2\sqrt{2}}\int\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right)dx$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}\,dx+\frac{\sqrt{2}}{2}\int\frac{1}{x^2+\sqrt{2}\,x+1}\,dx\right.$$

$$\left.-\frac{1}{2}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\,dx+\frac{\sqrt{2}}{2}\int\frac{1}{x^2-\sqrt{2}\,x+1}\,dx\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\int\frac{\left(x^2+\sqrt{2}\,x+1\right)'}{x^2+\sqrt{2}\,x+1}\,dx+\frac{\sqrt{2}}{2}\int\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx\right.$$

$$\left.-\frac{1}{2}\int\frac{\left(x^2-\sqrt{2}\,x+1\right)'}{x^2-\sqrt{2}\,x+1}\,dx+\frac{\sqrt{2}}{2}\int\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx\right)$$

$$= \frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\log\left(x^2+\sqrt{2}\,x+1\right)+\tan^{-1}\left(\sqrt{2}\,x+1\right)\right.$$

$$\left.-\frac{1}{2}\log\left(x^2-\sqrt{2}\,x+1\right)\tan^{-1}\left(\sqrt{2}\,x-1\right)\right\}$$

$$= \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}+\frac{1}{2\sqrt{2}}\left\{\tan^{-1}\left(\sqrt{2}\,x+1\right)+\tan^{-1}\left(\sqrt{2}\,x-1\right)\right\}.$$

問題 8.2

(1) $t=\tan\dfrac{x}{2}$ とおけば

$$\int\frac{dx}{\sin x}=\int\frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}\,dt=\int\frac{dt}{t}=\log|\,t\,|=\log\left|\frac{1-\cos x}{\sin x}\right|.$$

(2) $t=\tan\dfrac{x}{2}$ とおけば

$$\int\frac{dx}{\sin x-\cos x} = \int\frac{1}{\dfrac{2t}{1+t^2}-\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int\frac{2}{t^2+2t-1}\,dt$$

$$= 2\int\frac{dt}{(t+1)^2-2} = 2\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right|$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{1-\cos x+\left(1-\sqrt{2}\,\right)\sin x}{1-\cos x+\left(1+\sqrt{2}\,\right)\sin x}\right|.$$

問題 8.3

(1) $t=\sqrt{x+1}$ とおけば

$$x=t^2-1\quad,\quad dx=2t\,dt\,.$$

よって

$$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2t}{t(\,t^2-1\,)}\,dt=2\int\frac{dt}{t^2-1}=\log\left|\frac{t-1}{t+1}\right|=\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|.$$

(2) $t=\sqrt{\dfrac{1-x}{1+x}}$ とおけば $t^2=\dfrac{1-x}{1+x}$ であるから

$$x=\frac{1-t^2}{1+t^2}\quad,\quad dx=-\frac{4t}{(\,1+t^2\,)^2}\,dt\,.$$

よって

$$
\begin{aligned}
\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx &= \int \frac{1+t^2}{1-t^2}\cdot t\cdot\left(-\frac{4t}{(1+t^2)^2}\right)dt = -\int \frac{4t^2}{(1+t^2)(1-t^2)}\,dt \\
&= 2\int\left(\frac{1}{1+t^2}-\frac{1}{1-t^2}\right)dt \\
&= 2\tan^{-1}t+\log\left|\frac{1-t}{1+t}\right| \\
&= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}+\log\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| .
\end{aligned}
$$

問題 8.4

$t=\sqrt{x^2+x+1}+x$ とおけば $t-x=\sqrt{x^2+x+1}$ だから, この両辺を 2 乗して

$$
t^2-2tx+x^2=x^2+x+1 \qquad \therefore \quad x=\frac{t^2-1}{2t+1} .
$$

これより

$$
\frac{dx}{dt}=\frac{2t(2t+1)-(t^2-1)\cdot 2}{(2t+1)^2}=\frac{2(t^2+t+1)}{(2t+1)^2} \qquad \therefore \quad dx=\frac{2(t^2+t+1)}{(2t+1)^2} .
$$

また,

$$
\sqrt{x^2+x+1}=t-x=t-\frac{t^2-1}{2t+1}=\frac{t(2t+1)-(t^2-1)}{2t+1}=\frac{t^2+t+1}{2t+1} .
$$

よって

$$
\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+x+1}}\,dx &= \int \frac{2t+1}{t^2-1}\cdot\frac{2t+1}{t^2+t+1}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}\,dt = 2\int\frac{dt}{t^2-1} \\
&= \log\left|\frac{t-1}{t+1}\right| = \log\left|\frac{\sqrt{x^2+x+1}+x-1}{\sqrt{x^2+x+1}+x+1}\right| .
\end{aligned}
$$

問題 8.5

(1) まずは定義域を調べると

$$
x-x^2>0 \iff x(x-1)<0 \iff 0<x<1
$$

であるから

$$
\sqrt{x-x^2}=\sqrt{x(1-x)}=x\sqrt{\frac{1-x}{x}} .
$$

そこで $t=\sqrt{\dfrac{1-x}{x}}$ とおけば

$$x = \frac{1}{t^2+1} \quad , \quad \sqrt{x - x^2} = \frac{t}{t^2+1} \quad , \quad dx = -\frac{2t}{(\,t^2+1\,)^2}\,dt$$

なので

$$\begin{aligned}\int \frac{dx}{\sqrt{x-x^2}} &= \int \frac{t^2+1}{t} \cdot \left(-\frac{2t}{(\,t^2+1\,)^2}\right) dt \;=\; -2\int \frac{dt}{t^2+1} \\ &= -2\tan^{-1} t \;=\; -2\tan^{-1}\sqrt{\frac{1-x}{x}}\ .\end{aligned}$$

(2) 定義域を調べると

$$a^2 - x^2 \geqq 0 \iff (\,x+a\,)(\,x-a\,) \leqq 0 \iff -a \leqq x \leqq a$$

であるから

$$\sqrt{a^2-x^2} = \sqrt{(\,a+x\,)(\,a-x\,)} = (\,a-x\,)\sqrt{\frac{a+x}{a-x}}\ .$$

そこで $t = \sqrt{\dfrac{a+x}{a-x}}$ とおけば

$$x = \frac{a(\,t^2-1\,)}{t^2+1} \quad , \quad \sqrt{a^2-x^2} = \frac{2at}{t^2+1} \quad , \quad dx = \frac{4at}{(\,t^2+1\,)^2}\,dt\ .$$

従って

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = \int \frac{(\,t^2+1\,)^2}{a^2(\,t^2-1\,)^2} \cdot \frac{t^2+1}{2at} \cdot \frac{4at}{(\,t^2+1\,)^2}\,dt \;=\; \frac{1}{a^2}\int \frac{2(\,t^2+1\,)}{(\,t^2-1\,)^2}\,dt\ .$$

ここで

$$\begin{aligned}\frac{2(\,t^2+1\,)}{(\,t^2-1\,)^2} &= \frac{2}{t^2-1} + \frac{4}{(\,t^2-1\,)^2} \;=\; \frac{2}{t^2-1} + \left(\frac{1}{t-1} - \frac{1}{t+1}\right)^2 \\ &= \frac{1}{(\,t-1\,)^2} + \frac{1}{(\,t+1\,)^2}\end{aligned}$$

だから

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{a^2-x^2}} &= \frac{1}{a^2}\int \left\{ \frac{1}{(\,t-1\,)^2} + \frac{1}{(\,t+1\,)^2} \right\} dt \\ &= \frac{1}{a^2}\left(-\frac{1}{t-1} - \frac{1}{t+1}\right) \;=\; -\frac{2t}{a^2(\,t^2-1\,)} \;=\; -\frac{\sqrt{a^2-x^2}}{a^2 x}\ .\end{aligned}$$