『微分積分』 第 15 章 演習問題の解答

問題 15.1

(1) 合成関数の微分法により

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{2}{2x+y} \cdot (1+t)e^t + \frac{1}{2x+y} \cdot (1-t)e^{-t} .$$

ここで $t=1$ のとき $x=e$, $y=e^{-1}$ だから,

$$\left.\frac{dz}{dt}\right|_{t=1} = \frac{2}{2e+e^{-1}} \cdot 2e + \frac{1}{2e+e^{-1}} \cdot 0 = \frac{4e^2}{2e^2+1} .$$

(2) 合成関数の微分法により

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{1}{x} \cdot (e^t - e^{-t}) + \frac{1}{1+y^2} \cdot 2e^{2t} .$$

ここで $t=\log 2$ のとき $x = e^{\log 2} + e^{-\log 2} = 2 + \dfrac{1}{2} = \dfrac{5}{2}$, $y = e^{2\log 2} = 4$ であるから

$$\left.\frac{dz}{dt}\right|_{t=\log 2} = \frac{2}{5} \times \left(2 - \frac{1}{2}\right) + \frac{1}{1+16} \times 8 = \frac{3}{5} + \frac{8}{17} = \frac{91}{85} .$$

問題 15.2

(1) $\dfrac{dz}{dt} = f_x \cdot 2 + f_y \cdot (-3) = 2f_x - 3f_y$,

$$\begin{aligned}\frac{d^2z}{dt^2} &= 2\{(f_x)_x \cdot 2 + (f_x)_y \cdot (-3)\} - 3\{(f_y)_x \cdot 2 + (f_y)_y \cdot (-3)\} \\ &= 4f_{xx} - 6f_{xy} - 6f_{yx} + 9f_{yy} = 4f_{xx} - 12f_{xy} + 9f_{yy} .\end{aligned}$$

(2) $\dfrac{dz}{dt} = f_x \cdot (2t+1) + f_y \cdot (2t-1) = (2t+1)f_x + (2t-1)f_y$,

$$\begin{aligned}\frac{d^2z}{dt^2} &= 2f_x + (2t+1)\{(f_x)_x \cdot (2t+1) + (f_x)_y \cdot (2t-1)\} \\ &\qquad +2f_y + (2t-1)\{(f_y)_x \cdot (2t+1) + (f_y)_y \cdot (2t-1)\} \\ &= 2f_x + (2t+1)^2 f_{xx} + (2t+1)(2t-1)f_{xy} \\ &\qquad +2f_y + (2t-1)(2t+1)f_{yx} + (2t-1)^2 f_{yy} \\ &= (2t+1)^2 f_{xx} + 2(2t+1)(2t-1)f_{xy} + (2t-1)^2 f_{yy} + 2f_x + 2f_y .\end{aligned}$$

(3) $\dfrac{dz}{dt} = f_x \cdot (-\sin t) + f_y \cdot \cos t = -\sin t \cdot f_x + \cos t \cdot f_y$,

$$\begin{aligned}
\frac{d^2z}{dt^2} &= -\cos t \cdot f_x - \sin t \left\{ (f_x)_x \cdot (-\sin t) + (f_x)_y \cdot \cos t \right\} \\
&\qquad\qquad -\sin t \cdot f_y + \cos t \left\{ (f_y)_x \cdot (-\sin t) + (f_y)_y \cdot \cos t \right\} \\
&= -\cos t \cdot f_x + \sin^2 t \cdot f_{xx} - \sin t \cos t \cdot f_{xy} \\
&\qquad\qquad -\sin t \cdot f_y - \sin t \cos t \cdot f_{yx} + \cos^2 t \cdot f_{yy} \\
&= \sin^2 t \cdot f_{xx} - 2 \sin t \cos t \cdot f_{xy} + \cos^2 t \cdot f_{yy} - \cos t \cdot f_x - \sin t \cdot f_y \ .
\end{aligned}$$

問題 15.3

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$ の両辺をもう一回 t に関して微分すれば

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \left(\frac{\partial^2 z}{\partial x^2} \cdot \frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x} \cdot \frac{dy}{dt} \right) \frac{dx}{dt} + \frac{\partial z}{\partial x} \cdot \frac{d^2x}{dt^2} \\
&\qquad + \left(\frac{\partial^2 z}{\partial x \partial y} \cdot \frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2} \cdot \frac{dy}{dt} \right) \frac{dy}{dt} + \frac{\partial z}{\partial y} \cdot \frac{d^2y}{dt^2} \\
&= \frac{\partial^2 z}{\partial x^2} \left(\frac{dx}{dt} \right)^2 + \frac{\partial^2 z}{\partial y \partial x} \cdot \frac{dy}{dt} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial x} \cdot \frac{d^2x}{dt^2} \\
&\qquad + \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{dx}{dt} \cdot \frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2} \left(\frac{dy}{dt} \right)^2 + \frac{\partial z}{\partial y} \cdot \frac{d^2y}{dt^2} \\
&= \frac{\partial^2 z}{\partial x^2} \left(\frac{dx}{dt} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{dx}{dt} \cdot \frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2} \left(\frac{dy}{dt} \right)^2 + \frac{\partial z}{\partial x} \cdot \frac{d^2x}{dt^2} + \frac{\partial z}{\partial y} \cdot \frac{d^2y}{dt^2} \ .
\end{aligned}$$

問題 15.4

(1) $u = \dfrac{y}{x}$, $v = y$ とおけば $x = \dfrac{v}{u}$, $y = v$ なので

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{1}{u} + \frac{\partial z}{\partial y} \cdot 1 = \frac{x}{y} \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \\
&= \frac{1}{y} \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) = 0 \ .
\end{aligned}$$

従って z は v に依らないから, $u = \dfrac{y}{x}$ のみの関数である.

(2) $u = ax + by$, $v = y$ とおけば $x = \dfrac{u - by}{a} = \dfrac{u - bv}{a}$, $y = v$ なので

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \left(-\frac{b}{a} \right) + \frac{\partial z}{\partial y} \cdot 1 = -\frac{1}{a} \left(b \frac{\partial z}{\partial x} - a \frac{\partial z}{\partial y} \right) = 0 \ .$$

従って z は v に依らないから，$u = ax + by$ のみの関数である.

(3) $u = xy$, $v = y$ とおけば $x = \dfrac{u}{v}$, $y = v$ なので

$$
\begin{aligned}
\frac{\partial}{\partial v}\left(\frac{z}{x^3}\right) &= \frac{\partial}{\partial x}\left(\frac{z}{x^3}\right)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{z}{x^3}\right)\cdot\frac{\partial y}{\partial v} \\
&= \frac{z_x \cdot x^3 - z\cdot 3x^2}{x^6}\cdot\left(-\frac{u}{v^2}\right) + \frac{z_y}{x^3}\cdot 1 \\
&= \frac{z_x \cdot x^3 - z\cdot 3x^2}{x^6}\cdot\left(-\frac{x}{y}\right) + \frac{z_y}{x^3} = \frac{-xz_x + 3z + yz_y}{x^3 y} \\
&= -\frac{xz_x - yz_y - 3z}{x^3 y} = 0 .
\end{aligned}
$$

従って $\dfrac{z}{x^3}$ は v に依らないから，$u = xy$ のみの関数である.

問題 15.5

$x = u^2 + v^2$, $y = u - v$ より

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 2u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} , \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = 2v\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}
\end{aligned}
$$

だから

$$
\begin{aligned}
u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} &= u\left(2u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) + v\left(2v\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \\
&= 2(u^2 + v^2)\frac{\partial z}{\partial x} + (u - v)\frac{\partial z}{\partial y} = 2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} .
\end{aligned}
$$

問題 15.6

(1) 合成函数の微分法により

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} , \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \qquad \cdots \quad (\sharp)
\end{aligned}
$$

が成り立つので

$$
\begin{aligned}
\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} &= \left(\frac{\partial z}{\partial x}\right)^2 + (\,u+v\,)\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + uv\left(\frac{\partial z}{\partial y}\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + y\left(\frac{\partial z}{\partial y}\right)^2 .
\end{aligned}
$$

(2) (♯) の両辺を u について偏微分すれば

$$
\begin{aligned}
\frac{\partial^2 z}{\partial u\partial v} &= \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y\partial x}\cdot\frac{\partial y}{\partial u} + \frac{\partial z}{\partial y} + u\left(\frac{\partial^2 z}{\partial x\partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial u}\right) \\
&= \frac{\partial^2 z}{\partial x^2} + v\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial z}{\partial y} + u\frac{\partial^2 z}{\partial x\partial y} + uv\frac{\partial^2 z}{\partial y^2} \\
&= \frac{\partial^2 z}{\partial x^2} + (\,u+v\,)\frac{\partial^2 z}{\partial x\partial y} + uv\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} \\
&= \frac{\partial^2 z}{\partial x^2} + x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} .
\end{aligned}
$$

問題 15.7

(1) 合成関数の微分法により

$$
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \qquad \cdots \quad (\spadesuit)
$$

$$
\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad \cdots \quad (\clubsuit)
$$

だから

$$
\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 &= \left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right)^2 \\
&= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2\cos\theta\sin\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 ,
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{r}\cdot\frac{\partial z}{\partial \theta}\right)^2 &= \left(-\sin\theta\frac{\partial z}{\partial x} + \cos\theta\frac{\partial z}{\partial y}\right)^2 \\
&= \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\theta\sin\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 .
\end{aligned}
$$

従って

$$
\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial z}{\partial \theta}\right)^2 &= (\,\cos^2\theta + \sin^2\theta\,)\left(\frac{\partial z}{\partial x}\right)^2 + (\,\sin^2\theta + \cos^2\theta\,)\left(\frac{\partial z}{\partial y}\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 .
\end{aligned}
$$

(2) (♠) より

$$
\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\cos\theta\frac{\partial^2 z}{\partial x^2}+\sin\theta\frac{\partial^2 z}{\partial y\partial x}\right)+\sin\theta\left(\cos\theta\frac{\partial^2 z}{\partial x\partial y}+\sin\theta\frac{\partial^2 z}{\partial y^2}\right) \\
&= \cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}\ .
\end{aligned}
$$

また, (♣) より

$$
\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= -r\cos\theta\frac{\partial z}{\partial x}-r\sin\theta\left(-r\sin\theta\frac{\partial^2 z}{\partial x^2}+r\cos\theta\frac{\partial^2 z}{\partial y\partial x}\right) \\
&\qquad -r\sin\theta\frac{\partial z}{\partial y}+r\cos\theta\left(-r\sin\theta\frac{\partial^2 z}{\partial x\partial y}+r\cos\theta\frac{\partial^2 z}{\partial y^2}\right) \\
&= -r\cos\theta\frac{\partial z}{\partial x}+r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2}-r^2\sin\theta\cos\theta\frac{\partial^2 z}{\partial y\partial x} \\
&\qquad -r\sin\theta\frac{\partial z}{\partial y}-r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} \\
&= r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2}-2r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2}-r\cos\theta\frac{\partial z}{\partial x}-r\sin\theta\frac{\partial z}{\partial y}\ .
\end{aligned}
$$

従って

$$
\begin{aligned}
&\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial z}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 z}{\partial \theta^2} \\
&=\cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}+\frac{1}{r}\left(\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}\right) \\
&\quad +\frac{1}{r^2}\left(r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2}-2r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2}-r\cos\theta\frac{\partial z}{\partial x}-r\sin\theta\frac{\partial z}{\partial y}\right) \\
&=\cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}+\frac{1}{r}\cos\theta\frac{\partial z}{\partial x}+\frac{1}{r}\sin\theta\frac{\partial z}{\partial y} \\
&\quad +\sin^2\theta\frac{\partial^2 z}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 z}{\partial y\partial x}+\cos^2\theta\frac{\partial^2 z}{\partial y^2}-\frac{1}{r}\cos\theta\frac{\partial z}{\partial x}-\frac{1}{r}\sin\theta\frac{\partial z}{\partial y} \\
&=(\cos^2\theta+\sin^2\theta)\frac{\partial^2 z}{\partial x^2}+(\sin^2\theta+\cos^2\theta)\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}
\end{aligned}
$$

となる.