『微分積分』 第 19 章 演習問題の解答

問題 19.1

(1) $$\iint_A (3x^2+y^2)\,dxdy = \int_{-1}^{1}\left(\int_0^2 (3x^2+y^2)\,dx\right)dy = \int_{-1}^{1}\Big[\, x^3+xy^2 \,\Big]_{x=0}^{x=2}\,dy$$

$$= \int_{-1}^{1}(8+2y^2)\,dy = \left[\, 8y+\frac{2}{3}y^3 \,\right]_{-1}^{1} = \left(8+\frac{2}{3}\right)-\left(-8-\frac{2}{3}\right) = \frac{52}{3}\ .$$

(2) $$\iint_A e^{x+2y}\,dxdy = \iint_A e^x\cdot e^{2y}\,dxdy = \left(\int_{-1}^{1} e^x\,dx\right)\left(\int_0^1 e^{2y}\,dy\right) = \Big[\, e^x \,\Big]_{-1}^{1}\times\left[\,\frac{1}{2}e^{2y}\,\right]_0^1$$

$$= \left(e-\frac{1}{e}\right)\times\left(\frac{1}{2}e^2-\frac{1}{2}\right) = \frac{(e^2-1)^2}{2e}\ .$$

(3) $$\iint_A (x-y)^2\,dxdy = \int_0^1\left(\int_0^2 (x-y)^2\,dx\right)dy = \int_0^1\left[\,\frac{1}{3}(x-y)^3\,\right]_{x=0}^{x=2}\,dy$$

$$= \int_0^1\left\{\frac{1}{3}(2-y)^3+\frac{1}{3}y^3\right\}dy = \left[\,-\frac{1}{12}(2-y)^4+\frac{1}{12}y^4\,\right]_0^1$$

$$= \left(-\frac{1}{12}+\frac{1}{12}\right)-\left(-\frac{4}{3}+0\right) = \frac{4}{3}\ .$$

(4) $$\iint_A \frac{dxdy}{1+x+y} = \int_0^{e-1}\left(\int_0^{e-1}\frac{dx}{1+x+y}\right)dy$$

$$= \int_0^{e-1}\Big[\,\log|1+x+y|\,\Big]_{x=0}^{x=e-1}\,dy = \int_0^{e-1}\Big\{\log(y+e)-\log(y+1)\Big\}\,dy$$

$$= \Big[\,(y+e)\log(y+e)-(y+e)-(y+1)\log(y+1)+(y+1)\,\Big]_0^{e-1}$$

$$= \Big\{(2e-1)\log(2e-1)-(2e-1)-e\log e+e\Big\}-(e\log e-e-\log 1+1)$$

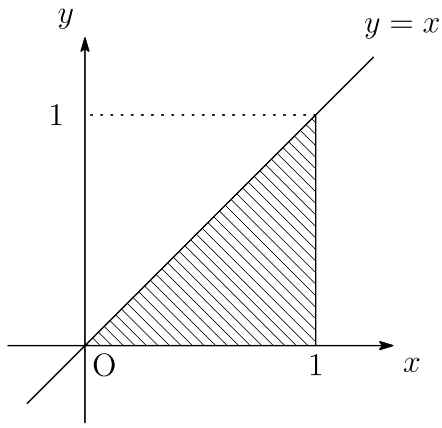
$$= (2e-1)\log(2e-1)-2e\ .$$

問題 19.2

(1) $$\iint_A (x^2+y^2)\,dxdy = \int_0^1 \left(\int_0^x (x^2+y^2)\,dy \right) dx$$

$$= \int_0^1 \left[x^2 y + \frac{1}{3} y^3 \right]_{y=0}^{y=x} dx = \int_0^1 \left(x^3 + \frac{1}{3} x^3 \right) dx$$

$$= \int_0^1 \frac{4}{3} x^3\,dx = \left[\frac{1}{3} x^4 \right]_0^1 = \frac{1}{3}\ .$$

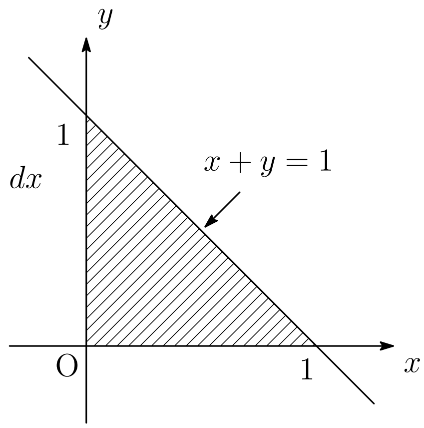


(2) $$\iint_A (x^2+y^2)\,dxdy = \int_0^1 \left(\int_0^{1-x} (x^2+y^2)\,dy \right) dx$$

$$= \int_0^1 \left[x^2 y + \frac{1}{3} y^3 \right]_{y=0}^{y=1-x} = \int_0^1 \left\{ x^2(1-x) + \frac{1}{3}(1-x)^3 \right\} dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4 \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} \right) - \left(-\frac{1}{12} \right)$$

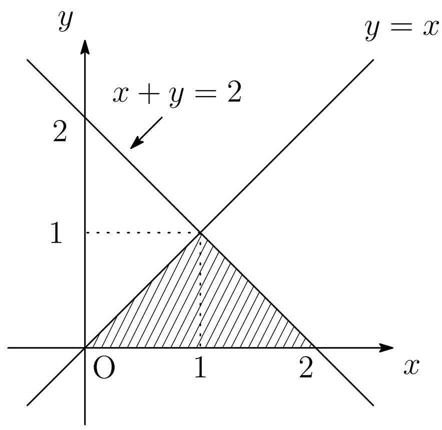
$$= \frac{1}{6}\ .$$



(3) $$\iint_A (x-y)\,dxdy = \int_0^1 \left(\int_y^{2-y} (x-y)\,dx \right) dy$$

$$= \int_0^1 \left[\frac{1}{2}(x-y)^2 \right]_{x=y}^{x=2-y} dy = \int_0^1 2(1-y)^2\,dy$$

$$= \left[-\frac{2}{3}(1-y)^3 \right]_0^1 = 0 - \left(-\frac{2}{3} \right) = \frac{2}{3}\ .$$



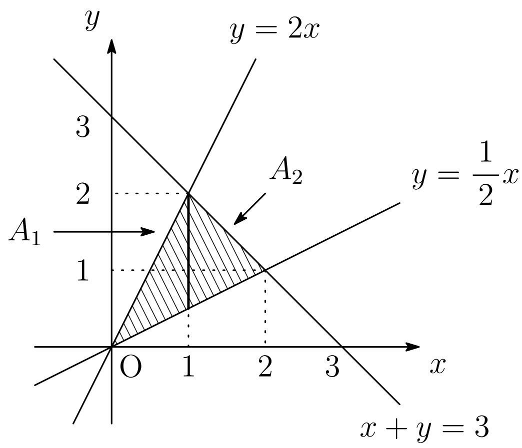
注意　領域 A を直線 $x=1$ によって 2 つに分け, それぞれの領域で積分したものを加える方法もある. 具体的には以下のようにする.

$$\iint_A (x-y)\,dxdy = \int_0^1 \left(\int_0^x (x-y)\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} (x-y)\,dy \right) dx$$

$$= \int_0^1 \left[-\frac{1}{2}(x-y)^2 \right]_{y=0}^{y=x} dx + \int_1^2 \left[-\frac{1}{2}(x-y)^2 \right]_{y=0}^{y=2-x} dx$$

$$= \int_0^1 \frac{1}{2}x^2\,dx + \int_1^2 \left\{ -2(x-1)^2 + \frac{1}{2}x^2 \right\} dx = \cdots$$

(4) 領域 A を直線 $x=1$ によって 2 つに分割し, 左側を A_1, 右側を A_2 とする.



$$\iint_A x\,dxdy = \iint_{A_1} x\,dxdy + \iint_{A_2} x\,dxdy$$

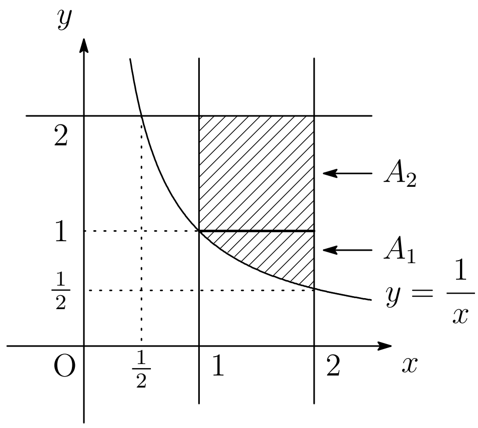
$$= \int_0^1 \left(\int_{\frac{1}{2}x}^{2x} x\,dy \right) dx + \int_1^2 \left(\int_{\frac{1}{2}x}^{3-x} x\,dy \right) dx$$

$$= \int_0^1 \Big[\, xy \,\Big]_{y=\frac{1}{2}x}^{y=2x} dx + \int_1^2 \Big[\, xy \,\Big]_{y=\frac{1}{2}x}^{y=3-x} dx$$

$$= \int_0^1 \frac{3}{2}x^2\,dx + \int_1^2 \left(3x - \frac{3}{2}x^2 \right) dx$$

$$= \left[\, \frac{1}{2}x^3 \,\right]_0^1 + \left[\, \frac{3}{2}x^2 - \frac{1}{2}x^3 \,\right]_1^2 = \left(\frac{1}{2} - 0 \right) + \left\{ (6-4) - \left(\frac{3}{2} - \frac{1}{2} \right) \right\} = \frac{3}{2}\ .$$

(5) 領域 A を直線 $y=1$ によって 2 つに分割し, 下側を A_1, 上側を A_2 とする.



$$\iint_A ye^{xy}\,dxdy = \iint_{A_1} ye^{xy}\,dxdy + \iint_{A_2} ye^{xy}\,dxdy$$

$$= \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{y}}^2 ye^{xy}\,dx \right) dy + \int_1^2 \left(\int_1^2 ye^{xy}\,dx \right) dy$$

$$= \int_{\frac{1}{2}}^1 \Big[\, e^{xy} \,\Big]_{x=\frac{1}{y}}^{x=2} dy + \int_1^2 \Big[\, e^{xy} \,\Big]_{x=1}^{x=2} dy$$

$$= \int_{\frac{1}{2}}^1 (\, e^{2y} - e \,)\,dy + \int_1^2 (\, e^{2y} - e^y \,)\,dy$$

$$= \left[\, \frac{1}{2}e^{2y} - ey \,\right]_{\frac{1}{2}}^1 + \left[\, \frac{1}{2}e^{2y} - e^y \,\right]_1^2$$

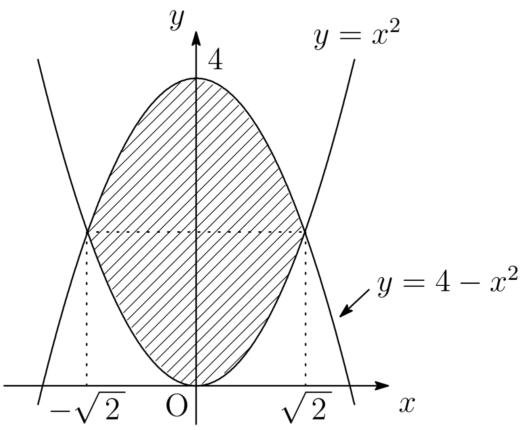
$$= \left(\frac{1}{2}e^2 - e \right) - \left(\frac{1}{2}e - \frac{1}{2}e \right) + \left(\frac{1}{2}e^4 - e^2 \right) - \left(\frac{1}{2}e^2 - e \right) = \frac{1}{2}e^4 - e^2\ .$$

注意 積分順序を反対にすれば一つの累次積分になるが, 計算はむしろたいへんになる.

$$\iint_A ye^{xy}\,dxdy \;=\; \int_1^2 \left(\int_{\frac{1}{x}}^2 ye^{xy}\,dy \right) dx \;=\; \cdots \;=\; \int_1^2 \left(\frac{2}{x}e^{2x} - \frac{1}{x^2}e^{2x} \right) dx$$

$$= \left[\, \frac{1}{x}e^{2x} \,\right]_1^2 \;=\; \frac{1}{2}e^4 - e^2\ .$$

(6) 2 つの放物線 $y = x^2$ と $y = 4 - x^2$ の交点の x 座標は, 方程式 $x^2 = 4 - x^2$ を解いて $x = \pm\sqrt{2}$.



$$\iint_A \sqrt{x^2+y}\,dxdy = \int_{-\sqrt{2}}^{\sqrt{2}}\left(\int_{x^2}^{4-x^2}\sqrt{x^2+y}\,dy\right)dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\left[\,\frac{2}{3}(\,x^2+y\,)^{\frac{3}{2}}\,\right]_{y=x^2}^{y=4-x^2} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\left\{\,\frac{2}{3}\cdot 4^{\frac{3}{2}} - \frac{2}{3}(\,2x^2\,)^{\frac{3}{2}}\,\right\}dx$$

$$\overset{(*)}{=} \int_{-\sqrt{2}}^{0}\left(\frac{16}{3}+\frac{4\sqrt{2}}{3}x^3\right)dx + \int_{0}^{\sqrt{2}}\left(\frac{16}{3}-\frac{4\sqrt{2}}{3}x^3\right)dx$$

$$= \left[\,\frac{16}{3}x + \frac{\sqrt{2}}{3}x^4\right]_{-\sqrt{2}}^{0} + \left[\,\frac{16}{3}x - \frac{\sqrt{2}}{3}x^4\right]_{0}^{\sqrt{2}}$$

$$= 0 - \left(-\frac{16\sqrt{2}}{3}+\frac{4\sqrt{2}}{3}\right) + \left(\frac{16\sqrt{2}}{3}-\frac{4\sqrt{2}}{3}\right) - 0 = 8\sqrt{2}\ .$$

注意 $(*)$ を付けた部分では,

$$(\,2x^2\,)^{\frac{3}{2}} = 2\sqrt{2}\,|\,x\,|^3 = \begin{cases} 2\sqrt{2}\,x^3 & \cdots\quad x \geqq 0 \text{ のとき} \\ -2\sqrt{2}\,x^3 & \cdots\quad x \leqq 0 \text{ のとき} \end{cases}$$

であることに着目して積分区間を 2 つに分けた.