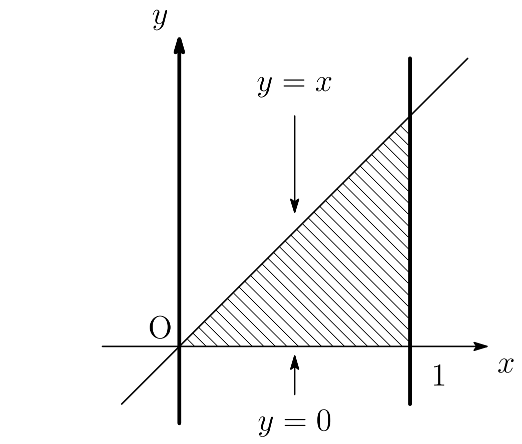
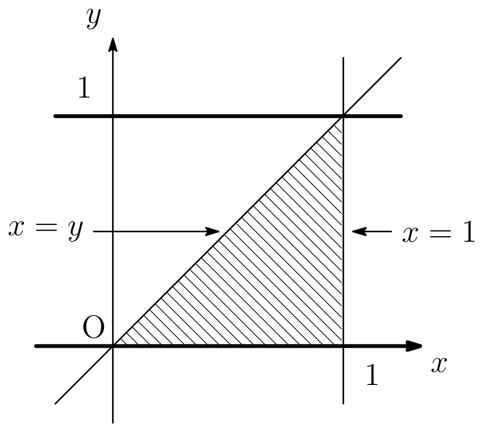
『微分積分』 第 20 章 演習問題の解答

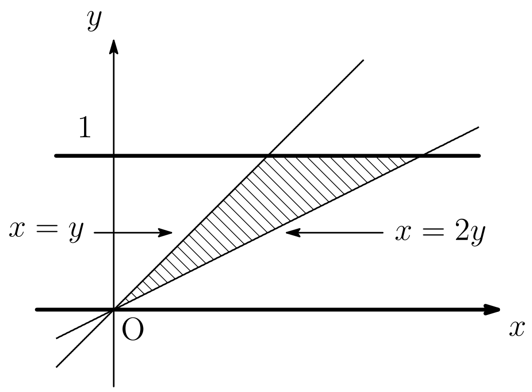
問題 20.1

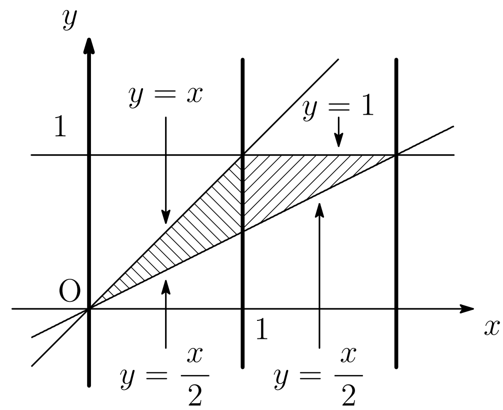
(1) $\displaystyle \int_0^1 \left(\int_0^x f(x\,,\,y)\,dy \right) dx = \int_0^1 \left(\int_y^1 f(x\,,\,y)\,dx \right) dy\ .$



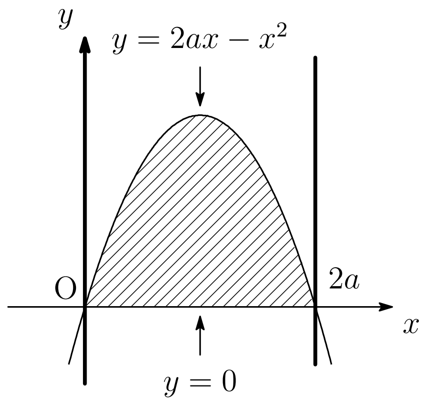


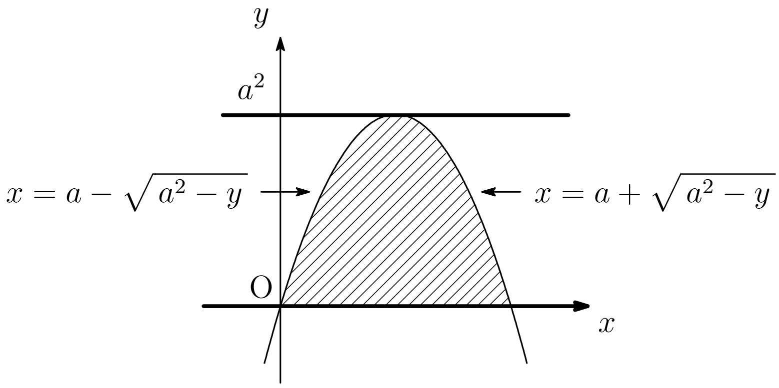
(2) $\displaystyle \int_0^1 \left(\int_y^{2y} f(x\,,\,y)\,dx \right) dy = \int_0^1 \left(\int_{\frac{1}{2}x}^{x} f(x\,,\,y)\,dy \right) dx + \int_1^2 \left(\int_{\frac{1}{2}x}^{1} f(x\,,\,y)\,dy \right) dx\ .$



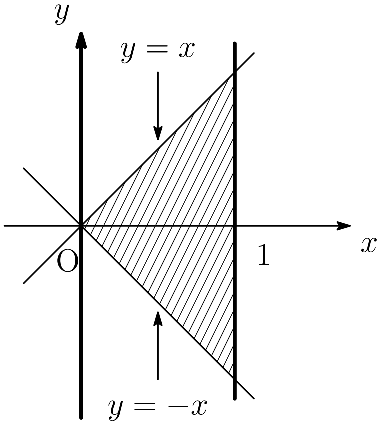


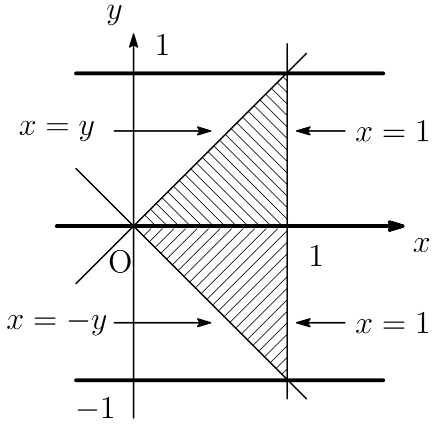
(3) $\displaystyle \int_0^{2a} dx \int_0^{2ax-x^2} f(x\,,\,y)\,dy = \int_0^{a^2} dy \int_{a-\sqrt{a^2-y}}^{a+\sqrt{a^2-y}} f(x\,,\,y)\,dx\ .$





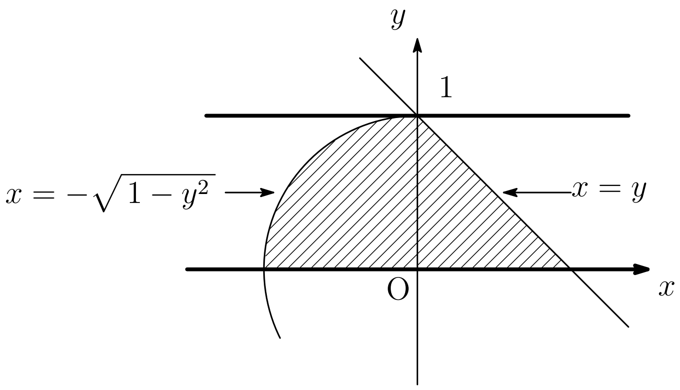
(4) $\displaystyle \int_0^1 dx \int_{-x}^{x} f(x\,,\,y)\,dy = \int_{-1}^{0} dy \int_{-y}^{1} f(x\,,\,y)\,dx + \int_0^1 dy \int_y^1 f(x\,,\,y)\,dx\ .$

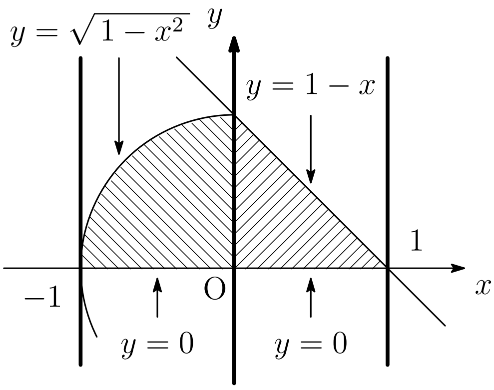




(5) $\displaystyle \int_0^1 \left(\int_{-\sqrt{1-y^2}}^{1-y} f(x\,,\,y)\,dx \right) dy$

$$= \int_{-1}^{0} \left(\int_0^{\sqrt{1-x^2}} f(x\,,\,y)\,dy \right) dx + \int_0^1 \left(\int_0^{1-x} f(x\,,\,y)\,dy \right) dx\ .$$



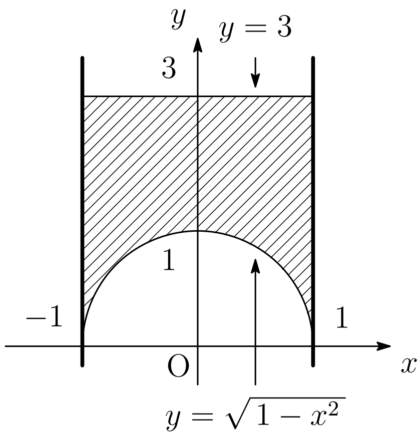


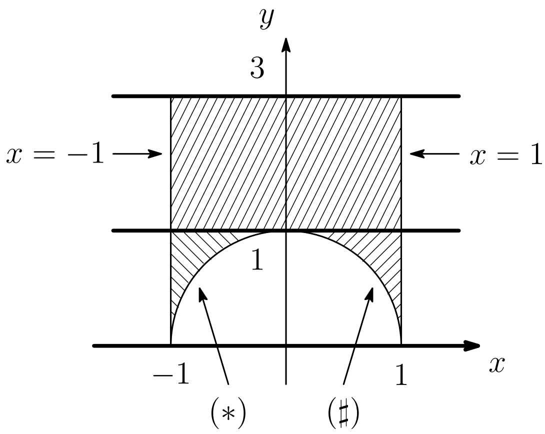
(6) $\displaystyle \int_{-1}^{1} \left(\int_{\sqrt{1-x^2}}^{3} f(x\,,\,y)\,dy \right) dx$

$$= \int_0^1 \left(\int_{-1}^{-\sqrt{1-y^2}} f(x\,,\,y)\,dx \right) dy + \int_0^1 \left(\int_{\sqrt{1-y^2}}^{1} f(x\,,\,y)\,dx \right) dy$$

$$+ \int_1^3 \left(\int_{-1}^{1} f(x\,,\,y)\,dx \right) dy\ .$$

(図は次ページ)





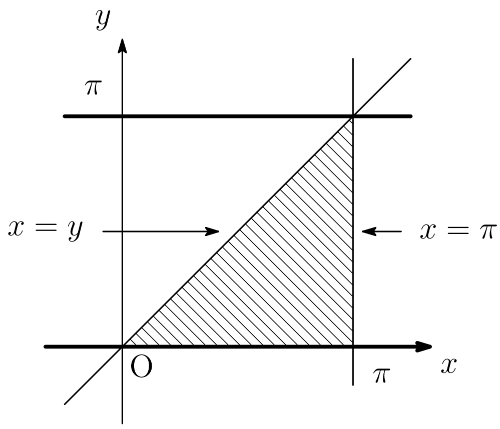
(∗) : $x = -\sqrt{1-y^2}$

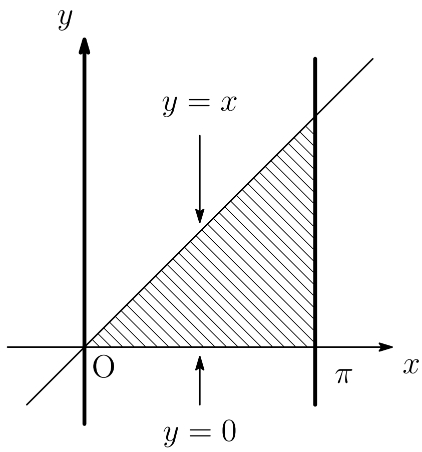
(♯) : $x = \sqrt{1-y^2}$

問題 20.2

(1) 積分順序を交換して

$$\int_0^\pi \left(\int_y^\pi \frac{y\sin x}{x}\,dx\right) dy = \int_0^\pi \left(\int_0^x \frac{y\sin x}{x}\,dy\right) dx = \int_0^\pi \left[\ \frac{y^2\sin x}{2x}\ \right]_{y=0}^{y=x} dx$$
$$= \frac{1}{2}\int_0^\pi x\sin x\,dx = \frac{1}{2}\left(\Big[-x\cos x\Big]_0^\pi + \int_0^\pi \cos x\,dx\right)$$
$$= \frac{\pi}{2} + \frac{1}{2}\Big[\sin x\Big]_0^\pi = \frac{\pi}{2}\ .$$





(2) 積分順序を交換して

$$\int_0^1 dy \int_{\sqrt[3]{y}}^1 e^{-x^2}\,dx = \int_0^1 dx \int_0^{x^3} e^{-x^2}\,dy = \int_0^1 \Big[\ e^{-x^2}y\ \Big]_{y=0}^{y=x^3} dx = \int_0^1 x^3e^{-x^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 x^2\cdot(-x^2)'e^{-x^2}\,dx = -\frac{1}{2}\left(\Big[\ x^2e^{-x^2}\ \Big]_0^1 - \int_0^1 2xe^{-x^2}\,dx\right)$$
$$= -\frac{1}{2}\left(\frac{1}{e} - \Big[-e^{-x^2}\Big]_0^1\right) = -\frac{1}{2}\left(\frac{1}{e}+\frac{1}{e}-1\right) = \frac{1}{2}-\frac{1}{e}\ .$$