# 『微分積分』 第 21 章 演習問題の解答

## 問題 21.1

(1) $u = x + y$, $v = x - y$ とおくと, $uv$ 平面の領域

$$B : 0 \leqq u \leqq 1 \text{ かつ } 0 \leqq v \leqq 2$$

が $xy$ 平面の領域 $A$ に対応し, $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$ であるから

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} .$$

従って

$$\begin{aligned} \iint_A x\,dx\,dy &= \iint_B \frac{1}{2}(u+v) \cdot \left| -\frac{1}{2} \right| dudv = \frac{1}{4} \iint_B (u+v)\,dudv \\ &= \frac{1}{4} \int_0^1 \left( \int_0^2 (u+v)\,dv \right) du = \frac{1}{4} \int_0^1 \left[ uv + \frac{1}{2} v^2 \right]_{v=0}^{v=2} du \\ &= \frac{1}{4} \int_0^1 (2u + 2)\,du = \frac{1}{4} \left[ u^2 + 2u \right]_0^1 = \frac{3}{4} . \end{aligned}$$

(2) $u = 2x + y$, $v = 2x - y$ とおくと, $uv$ 平面の領域

$$B : |u| \leqq 1 \text{ かつ } |v| \leqq 1$$

が $xy$ 平面の領域 $A$ に対応し, $x = \frac{1}{4}(u + v)$, $y = \frac{1}{2}(u - v)$ であるから

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4} .$$

従って

$$\begin{aligned} \iint_A (2x - 3y)\,dx\,dy &= \iint_B (-u + 2v) \cdot \left| -\frac{1}{4} \right| dudv = \frac{1}{4} \iint_B (-u + 2v)\,dudv \\ &= \frac{1}{4} \int_{-1}^1 \left( \int_{-1}^1 (-u + 2v)\,dv \right) du = \frac{1}{4} \int_{-1}^1 \left[ -uv + v^2 \right]_{v=-1}^{v=1} du \\ &= \frac{1}{4} \int_{-1}^1 (-2u)\,du = \frac{1}{4} \left[ -u^2 \right]_{-1}^1 = 0 . \end{aligned}$$

**注意** (2) で $u = |2x + y|$, $v = |2x - y|$ などとおいてはいけない. ( その後の処理ができない. )

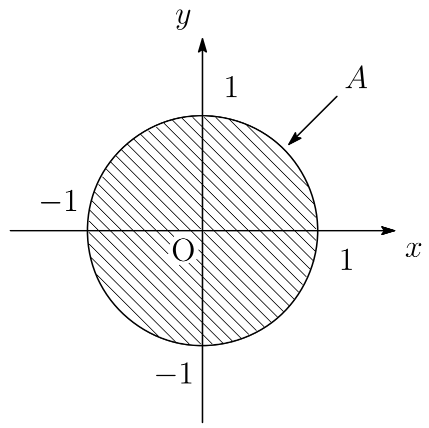
**問題 21.2**

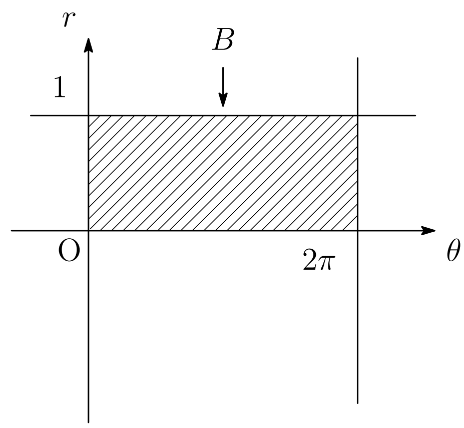
(1) $x = r\cos\theta$, $y = r\sin\theta$ とおくと，$r\theta$ 平面の領域

$$B : 0 \leqq \theta \leqq 2\pi \text{ かつ } 0 \leqq r \leqq 1$$

が $xy$ 平面の領域 $A$ に対応する．従って

$$\iint_A (x^2 + y^2)\,dx\,dy = \iint_B r^3\,drd\theta = \left(\int_0^1 r^3\,dr\right) \times \left(\int_0^{2\pi} d\theta\right) = \frac{1}{4} \times 2\pi = \frac{\pi}{2}\ .$$



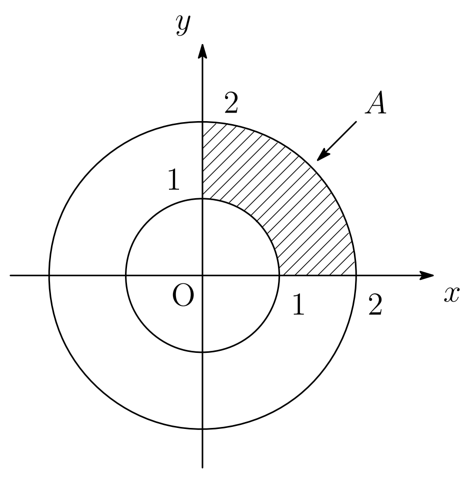



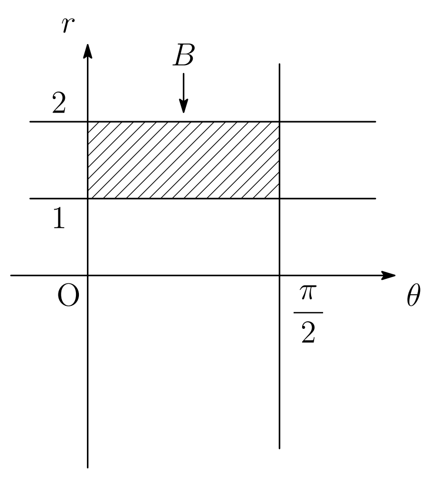

(2) $x = r\cos\theta$, $y = r\sin\theta$ とおくと，$r\theta$ 平面の領域

$$B : 0 \leqq \theta \leqq \frac{\pi}{2} \text{ かつ } 1 \leqq r \leqq 2$$

が $xy$ 平面の領域 $A$ に対応する．従って

$$\begin{aligned}\iint_A xy\,dx\,dy &= \iint_B r^3 \cos\theta \sin\theta\,drd\theta = \left(\int_1^2 r^3\,dr\right) \times \left(\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2\theta\,d\theta\right) \\ &= \left[\frac{1}{4}r^4\right]_1^2 \times \left[-\frac{1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{15}{4} \times \left\{\frac{1}{4} - \left(-\frac{1}{4}\right)\right\} = \frac{15}{8}\ .\end{aligned}$$



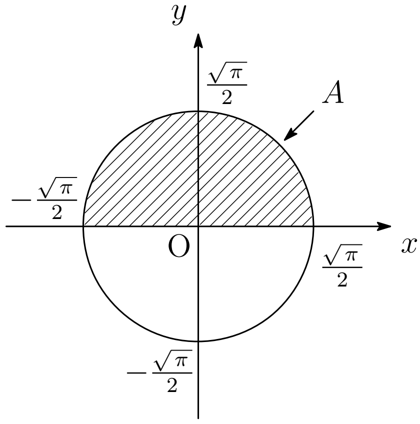



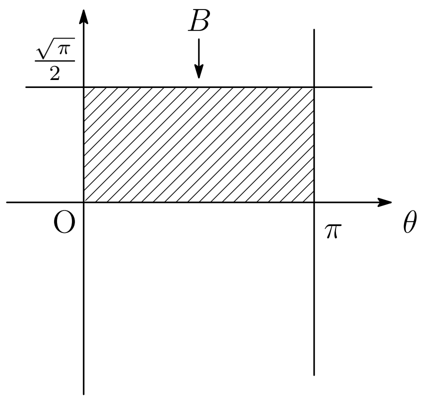
(3) $x = r\cos\theta\,,\ y = r\sin\theta$ とおくと, $r\theta$ 平面の領域

$$B\ :\ 0 \leqq \theta \leqq \pi \ \text{かつ}\ 0 \leqq r \leqq \frac{\sqrt{\pi}}{2}$$

が $xy$ 平面の領域 $A$ に対応する. 従って

$$\begin{aligned}
\iint_A \sin(\,x^2+y^2\,)\,dx\,dy &= \iint_B \sin r^2 \cdot r\,drd\theta = \left(\int_0^{\frac{\sqrt{\pi}}{2}} r\sin r^2\,dr\right)\left(\int_0^{\pi} d\theta\right) \\
&= \left[-\frac{1}{2}\cos r^2\right]_0^{\frac{\sqrt{\pi}}{2}} \times \left[\,\theta\,\right]_0^{\pi} \\
&= \left(-\frac{1}{2\sqrt{2}} + \frac{1}{2}\right) \times \pi = \frac{(\,2-\sqrt{2}\,)\pi}{4}\ .
\end{aligned}$$




(4) $x = r\cos\theta\,,\ y = r\sin\theta$ とおくと, $r\theta$ 平面の領域

$$B\ :\ 0 \leqq \theta \leqq \frac{\pi}{2}\ \text{かつ}\ 0 \leqq r \leqq \cos\theta$$

が $xy$ 平面の領域 $A$ に対応する. 従って

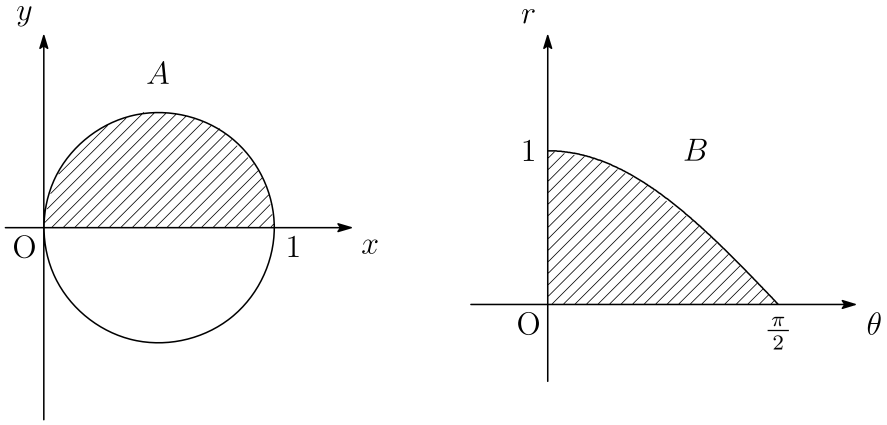
$$\iint_A \sqrt{1-x^2-y^2}\,dxdy = \iint_B \sqrt{1-r^2}\cdot r\,drd\theta = \int_0^{\frac{\pi}{2}}\left(\int_0^{\cos\theta} r\cdot(\,1-r^2\,)^{\frac{1}{2}}\,dr\right)d\theta$$

となる. ここで

$$\begin{aligned}
\int_0^{\cos\theta} r\cdot(\,1-r^2\,)^{\frac{1}{2}}\,dr &= -\frac{1}{2}\int_0^{\cos\theta}(\,1-r^2\,)^{\frac{1}{2}}\cdot(\,1-r^2\,)'dr = \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_{r=0}^{r=\cos\theta} \\
&= \frac{1}{3}(\,1-\sin^3\theta\,) = \frac{1}{3}(\,1-\sin^2\theta\,\sin\theta\,) \\
&= \frac{1}{3}\{\,1-\sin\theta(\,1-\cos^2\theta\,)\}
\end{aligned}$$

だから

$$\iint_A \sqrt{1-x^2-y^2}\,dxdy = \frac{1}{3}\left[\theta + \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{3}\left\{\frac{\pi}{2} - \left(1 - \frac{1}{3}\right)\right\} = \frac{\pi}{6} - \frac{2}{9}\,.$$

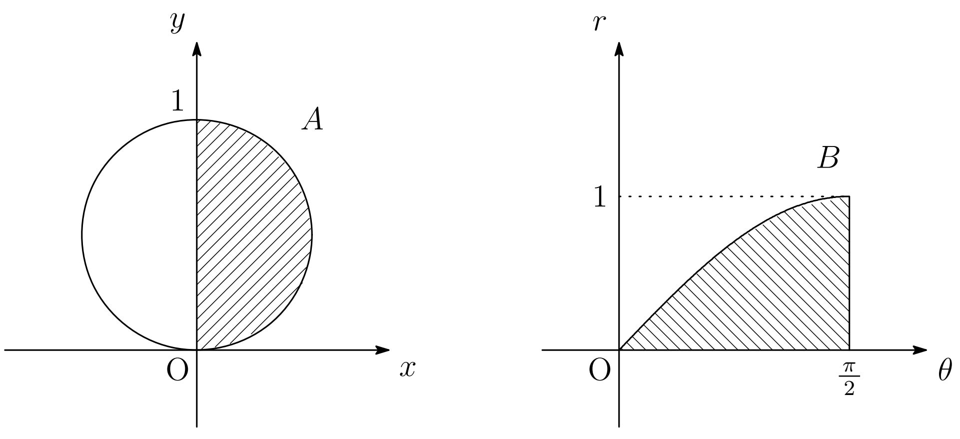



(5) $x = r\cos\theta$, $y = r\sin\theta$ とおくと, $r\theta$ 平面の領域

$$B\ :\ 0 \leqq \theta \leqq \frac{\pi}{2}\ \ \text{かつ}\ \ 0 \leqq r \leqq \sin\theta$$

が $xy$ 平面の領域 $A$ に対応する. 従って

$$\iint_A xy\,dxdy = \iint_B r^3\cos\theta\sin\theta\,drd\theta = \int_0^{\frac{\pi}{2}}\left(\int_0^{\sin\theta} r^3\cos\theta\sin\theta\,dr\right)d\theta$$
$$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_{r=0}^{r=\sin\theta} d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}}\cos\theta\sin^5\theta\,d\theta$$
$$= \frac{1}{4}\left[\frac{1}{6}\sin^6\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{24}\,.$$



(6) $x = r\cos\theta\,,\ y = r\sin\theta$ とおくと, $r\theta$ 平面の領域

$$B\ :\ -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{4}\ \ \text{かつ}\ \ 0 \leqq r \leqq \cos\theta$$

が $xy$ 平面の領域 $A$ に対応する. 従って

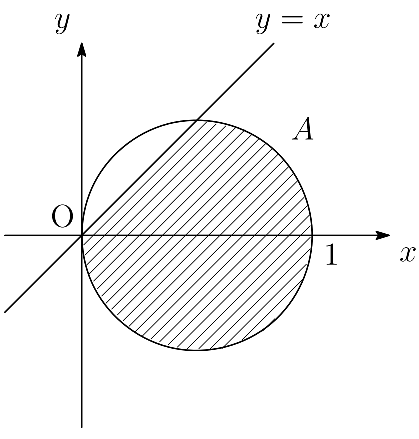
$$\begin{aligned}\iint_A (x^2+y^2)\,dxdy &= \iint_B r^3\,drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left(\int_0^{\cos\theta} r^3\,dr\right) d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left[\ \frac{1}{4}r^4\ \right]_{r=0}^{r=\cos\theta} d\theta = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^4\theta\,d\theta\ .\end{aligned}$$

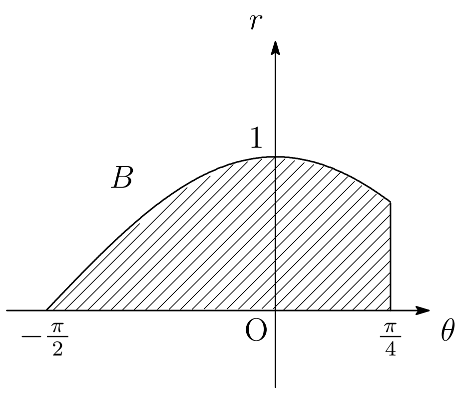
ここで

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

だから

$$\begin{aligned}\iint_A (x^2+y^2)\,dxdy &= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right) d\theta \\ &= \frac{1}{4}\left[\ \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\ \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \\ &= \frac{1}{4}\left(\frac{3\pi}{32} + \frac{1}{4}\right) - \frac{1}{4}\left(-\frac{3\pi}{16}\right) = \frac{9\pi}{128} + \frac{1}{16}\ .\end{aligned}$$



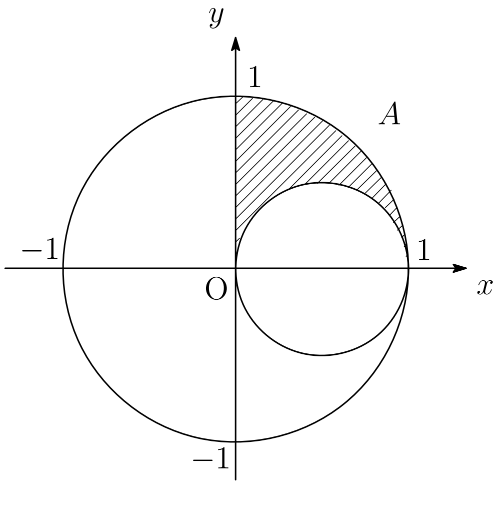



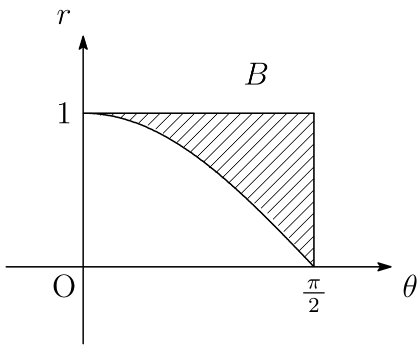
(7) $x = r\cos\theta\,,\ y = r\sin\theta$ とおくと, $r\theta$ 平面の領域

$$B\ :\ 0 \leqq \theta \leqq \frac{\pi}{2}\ \text{ かつ }\ \cos\theta \leqq r \leqq 1$$

が $xy$ 平面の領域 $A$ に対応する. 従って

$$\begin{aligned}
\iint_A x\,dxdy &= \iint_B r^2\cos\theta\,drd\theta = \int_0^{\frac{\pi}{2}}\left(\int_{\cos\theta}^1 r^2\cos\theta\,dr\right)d\theta \\
&= \int_0^{\frac{\pi}{2}}\left[\,\frac{1}{3}r^3\cos\theta\,\right]_{r=\cos\theta}^{r=1} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}}(\cos\theta - \cos^4\theta)\,d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}}\left(\cos\theta - \frac{3}{8} - \frac{1}{2}\cos 2\theta - \frac{1}{8}\cos 4\theta\right)d\theta \\
&= \frac{1}{3}\left[\,\sin\theta - \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta - \frac{1}{32}\sin 4\theta\,\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{3}\left(1 - \frac{3\pi}{16}\right) = \frac{1}{3} - \frac{\pi}{16}\ .
\end{aligned}$$



## 問題 21.3

(1) $u = \dfrac{x}{3}$ , $v = \dfrac{y}{2}$ とおけば $x = 3u$ , $y = 2v$ だから,

$$\frac{\partial(\,x\,,\,y\,)}{\partial(\,u\,,\,v\,)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6\ .$$

そして $uv$ 平面の領域

$$B\ :\ u^2 + v^2 \leqq 1 \quad \text{かつ} \quad u \geqq 0$$

が $xy$ 平面の領域 $A$ に対応する. 従って

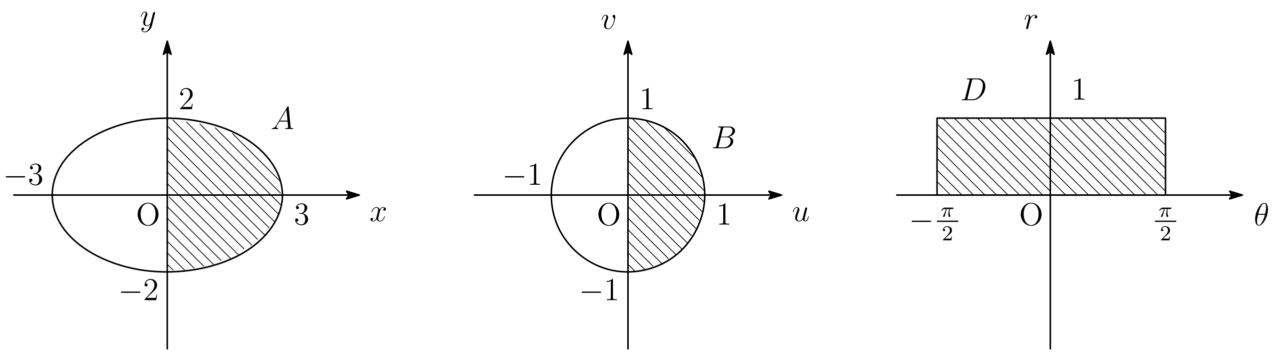
$$\iint_A xy^2\,dx\,dy = \iint_B 12uv^2 \cdot 6\,du\,dv = 72\iint_B uv^2\,du\,dv \qquad \cdots \quad (*)\ .$$

ここでさらに $u = r\cos\theta$ , $v = r\sin\theta$ とおけば, $r\theta$ 平面の領域

$$D\ :\ -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \quad \text{かつ} \quad 0 \leqq r \leqq 1$$

が $uv$ 平面の領域 $B$ に対応するから

$$\begin{aligned}(*) \;&=\; 72\iint_D r^4\cos\theta\,\sin^2\theta\,dr\,d\theta \;=\; 72\left(\int_0^1 r^4\,dr\right)\times\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\sin^2\theta\,d\theta\right)\\ &=\; 72\times\left[\,\frac{1}{5}r^4\,\right]_0^1\times\left[\,\frac{1}{3}\sin^3\theta\,\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \;=\; 72\times\frac{1}{5}\times\frac{2}{3} \;=\; \frac{48}{5}\ .\end{aligned}$$




**別解** $x = 3r\cos\theta$, $y = 2r\sin\theta$ とおけば, $r\theta$ 平面の領域

$$B\ :\ 0 \leqq r \leqq 1 \ \text{かつ}\ -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}$$

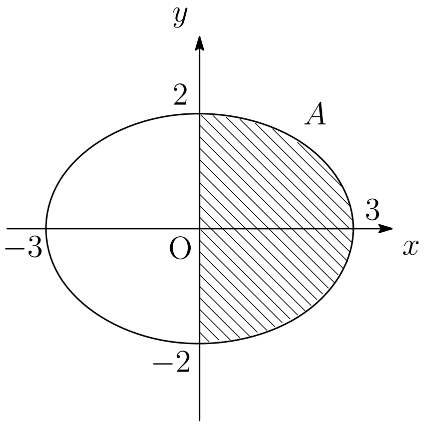
が $xy$ 平面の領域 $A$ に対応する. そして

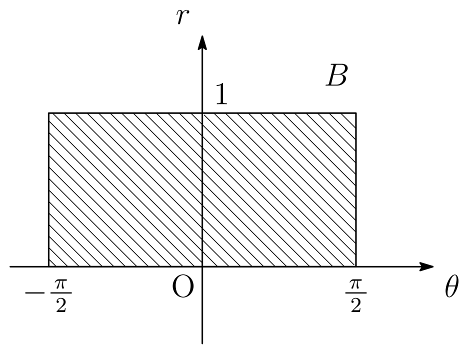
$$\frac{\partial(x\,,\,y)}{\partial(r\,,\,\theta)} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 6r\cos^2\theta + 6r\sin^2\theta = 6r$$

であるから

$$\iint_A xy^2\,dxdy = \iint_B 12r^3\cos\theta\sin^2\theta\cdot 6r\,drd\theta = 72\left(\int_0^1 r^4\,dr\right)\times\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\sin^2\theta\,d\theta\right)$$

$$= 72\times\left[\,\frac{1}{5}r^4\,\right]_0^1\times\left[\,\frac{1}{3}\sin^3\theta\,\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 72\times\frac{1}{5}\times\frac{2}{3} = \frac{48}{5}\,.$$

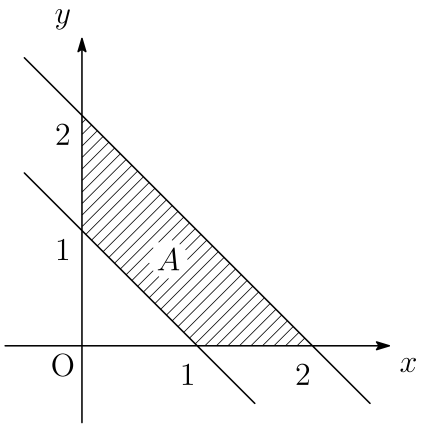


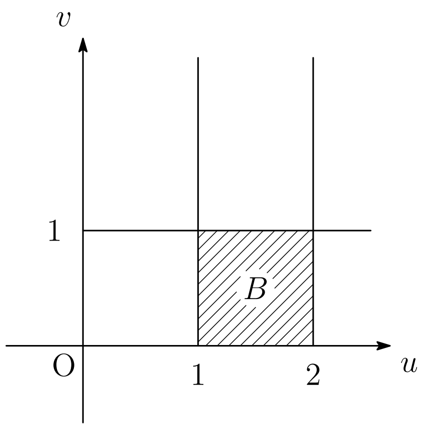




(2) $x+y=u$ , $y=uv$ とおけば $x=u(1-v)$, $y=uv$ であり, $uv$ 平面の領域

$$B\ :\ 1\leqq u\leqq 2\ \text{かつ}\ 0\leqq v\leqq 1$$

が $xy$ 平面の領域 $A$ に対応する.




そして

$$\frac{\partial(x\,,\,y)}{\partial(u\,,\,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v)+uv = u$$

であるから

$$\iint_A \frac{x^2+y^2}{(x+y)^3}\,dx\,dy = \iint_B \frac{u^2(1-v)^2+u^2v^2}{u^3}\cdot u\,du\,dv$$

$$= \iint_B\{\,(1-v)^2+v^2\,\}\,du\,dv = \iint_B(\,2v^2-2v+1\,)\,du\,dv$$

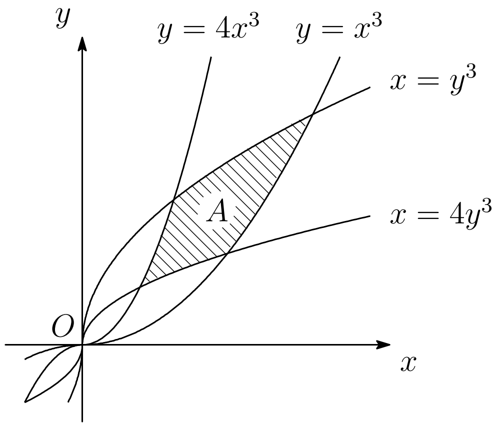
$$= \left( \int_1^2 du \right) \times \left( \int_0^1 (\, 2v^2 - 2v + 1\,)\, dv \right)$$

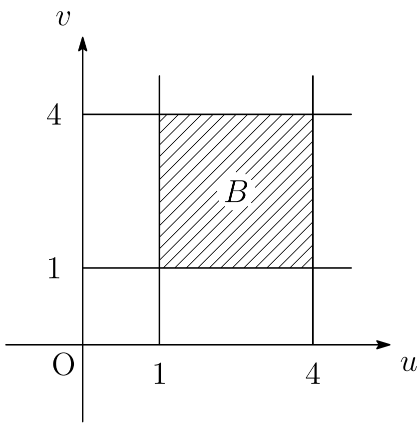
$$= \Big[\ u\ \Big]_1^2 \times \Big[\ \frac{2}{3}v^3 - v^2 + v\ \Big]_0^1 = \frac{2}{3}\ .$$

(3) $u = \dfrac{y}{x^3}$ , $v = \dfrac{x}{y^3}$ とおけば, $uv$ 平面の領域

$$B\ :\ 1 \leqq u \leqq 4\ \text{ かつ }\ 1 \leqq v \leqq 4$$

が $xy$ 平面の領域 $A$ に対応する.




そして

$$\frac{\partial(\,u\,,\,v\,)}{\partial(\,x\,,\,y\,)} = \begin{vmatrix} -\dfrac{3y}{x^4} & \dfrac{1}{x^3} \\ \dfrac{1}{y^3} & -\dfrac{3x}{y^4} \end{vmatrix} = \frac{9}{x^3y^3} - \frac{1}{x^3y^3} = \frac{8}{x^3y^3}$$

であることより

$$\frac{\partial(\,x\,,\,y\,)}{\partial(\,u\,,\,v\,)} = \frac{1}{\dfrac{\partial(\,x\,,\,y\,)}{\partial(\,u\,,\,v\,)}} = \frac{1}{8}x^3y^3\ .$$

ここで $uv = \dfrac{1}{x^2y^2}$ だから $xy = \dfrac{1}{\sqrt{uv}} = u^{-\frac{1}{2}}v^{-\frac{1}{2}}$ . よって $\dfrac{\partial(x\,,\,y)}{\partial(u\,,\,v)} = \dfrac{1}{8}u^{-\frac{3}{2}}v^{-\frac{3}{2}}$ . 従って

$$\iint_A dxdy \ = \ \iint_B \frac{1}{8}u^{-\frac{3}{2}}v^{-\frac{3}{2}}\, dudv \ = \ \frac{1}{8}\left( \int_1^4 u^{-\frac{3}{2}}\, du \right) \times \left( \int_1^4 v^{-\frac{3}{2}}\, dv \right)$$

$$= \ \frac{1}{8}\Big[\ -2u^{-\frac{1}{2}}\ \Big]_1^4 \times \Big[\ -2v^{-\frac{1}{2}}\ \Big]_1^4 \ = \ \frac{1}{8}\ .$$

**注意** 実を言うと領域 $A$ は原点 O も含んでいるが, 積分の値には関係ないので無視して良い.