Strategyproof Matching with Maximum and Minimum Quotas for Two Types of Members

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Abstract. This paper considers the problem of matching between laboratories and students when there are two types of students, such as domestic and overseas students. Each laboratory has minimum and maximum quotas for the number of overseas students and the total number of students. When there is only one type of student, a strategyproof matching algorithm was shown by Fragiadakis et.al. Their algorithm uses a precedence list (PL) to achieve fairness and nonwastefulness properties. This paper generalizes the algorithm for the cases when there are two types of students. Our algorithm achieves fairness based on the PL-list, nonwastefulness, and strategyproofness.

Keywords: matching algorithm · deferred acceptance · minimum quotas · maximum quotas

1 Introduction

Matching algorithms are widely discussed for many cases, such as between students and laboratories, between medical students and hospitals, between workers and farms, and so on. There are several surveys of matching algorithms [2, 3, 10, 18]. Stable matching between schools and students was discussed in [14, 17]. A protocol based on negotiation was shown in [16]. Matching under imperfect preferences was shown in [12].

In many cases, there is a maximum quota for the latter party (laboratories, hospitals, farms) that cannot be exceeded. In many real-world markets, there is a minimum quota that must be achieved. First, the minimum quota was introduced to open programs in a college [4]. For the assignment problem with the minimum quotas proposed by Kamada and Kojima [13] and several matching algorithms have been shown [7, 9, 19]. To obtain a matching that satisfies the minimum and maximum quotas, the MSDA algorithm was proposed [8], which uses a precedence list (PL) to achieve fairness and nonwastefulness properties. It achieves PL-fareness.

In many real-world matching situations, agents have multiple types. Examples of the types are gender, nationality, age, and so on. Two different cases were discussed. The first case is when an agent might simultaneously satisfy multiple types [15]. The other case is when each agent satisfies one type [5, 6]. This paper considers the latter case.

One of such problem settings is affirmative action [11]. Schools have minority reserve slots, and any minority applicant is preferred to any majority applicant until the slot is filled by minority students.

There are multiple types of student cases other than affirmative action. For example, overseas students need special care with language, thus, it is not good that some laboratories have many overseas students. If the number of male students is small, it is not desirable for some laboratories to have many male students. Such problems are not affirmative action, thus, preferential treatment in matching is not a good solution. Since the MSDA algorithm can assign students so that the maximum and minimum quotas are satisfied, using the MSDA algorithm multiple times might seem to be one solution. Set the minimum and maximum quota for each type of students (overseas/domestic, or female/male), and the matching is independently done for each type. Though the algorithm is simple, the number of assigned students might differ among laboratories: one laboratory is assigned the minimum number for both types, and another laboratory is assigned the maximum number for both types. This might produce disproportionate results. Thus, a better solution is to set the minimum and maximum quotas for the minor type of agents and the total number of agents.

This paper discusses the matching algorithm when the minimum and maximum quotas for the number of overseas students and the total number of students are set. We cannot use the MSDA algorithm for this problem, thus, we show a new algorithm for the problem. Section 2 defines the problem. Section 3 shows our new algorithm. Section 4 concludes the paper.,

2 Problem Definition

This paper models the problem as a matching market between laboratories and students. A market P consists of $P = (S, R, C, p.q, pr, qr, \succ_T, \succ_C)$. Let $S = \{s_1, s_2, \ldots, s_n\}$ and $R = \{r_1, r_2, \ldots, r_m\}$ be the set of domestic and overseas students, respectively. In this paper, a student refers to a domestic or international student. Let $T = S \cup R$ be the set of all students. Let $C = \{c_1, c_2, \ldots, c_l\}$ be the set of laboratories. Let $p = (p_{c_1}, p_{c_2}, \ldots, p_{c_l})$ and $q = (q_{c_1}, q_{c_2}, \ldots, q_{c_l})$ be the list of minimum and maximum quotas of students for laboratories, respectively. Let $pr = (pr_{c_1}, pr_{c_2}, \ldots, pr_{c_l})$ and $qr = (qr_{c_1}, qr_{c_2}, \ldots, qr_{c_l})$ be the list of minimum and maximum quotas of overseas students for laboratories, respectively. For each laboratory $c \in C$, $p_c \geq pr_c \geq 0$, $q_c \geq qr_c$, $q_c \geq p_c$, and $qr_c \geq pr_c$ must be satisfied. In addition, inequalities $\sum_{c \in C} q_c \geq n + m \geq \sum_{c \in C} p_c$, $\sum_{c \in C} qr_c \geq m \geq \sum_{c \in C} pr_c$, and $n \geq \sum_{c \in C} \max(0, p_c - qr_c)$ must be satisfied to have a matching that satisfies the maximum and minimum quotas. The last inequality exists because each laboratory must have at least $\max(0, p_c - qr_c)$ domestic students.

Each student $t \in T$ has a strict preference relation $\succ_t$ over C, respectively. Each laboratory c has a strict preference relation $\succ_c$ over T. Let $\succ_T = \{\succ_t | t \in T\}$ and $\succ_C = \{\succ_c | c \in C\}$.

Definition 1 *A matching is a mapping* $\mu : T \cup C \to 2^T \cup C$ *that satisfies the following properties.*

1. $\mu(t) \in C$ *for all* $t \in T$.
2. $\mu(c) \subseteq T$ *for all* $c \in C$.
3. *For any* $t \in T$ *and* $c \in C$, $\mu(t) = c$ *is satisfied if and only if* $t \in \mu(c)$.

Definition 2 *A matching* μ *is feasible if* $p_c \leq |\mu(c)| \leq q_c$ *and* $pr_c \leq |\mu(c) \cap R| \leq qr_c$ *for all* $c \in C$. *Let* $\mathcal{M}$ *be the set of feasible matchings.*

Definition 3 *A mechanism* $\chi : (\succ_T, \succ_C) \to \mathcal{M}$ *is a function that takes as input any possible preference profile of the students and laboratories and gives as an output a feasible matching.*

We write $\chi_i(\succ_T, \succ_C)$ for the assignment of agent $i \in T \cup C$.

Let $\succ'_t$ be any (false) preference for student $t \in T$. Let $(\succ'_t, \succ_{T \setminus \{t\}})$ be the tuple of preferences where student t's preference is changed from true $\succ_t$ to $\succ'_t$.

Definition 4 *A mechanism* χ *is strategyproof if* $\chi_t(\succ_T, \succ_C) \succeq_t \chi_t(\succ'_t, \succ_{T \setminus \{t\}}, \succ_C)$ *for all* $\succ_T, t \in T$, $\succ_C$, *and* $\succ'_t$.

A mechanism is strategyproof if no student has any incentive to misreport his/her preference.

To solve the problem, we introduce a precedence list (PL) $\succ_{PL}$ that ranks all students as in [8]. One example of a precedence list is the GPA score of the students. This paper assumes that S and R are sorted by PL, that is, $s_1 \succ_{PL} s_2 \succ_{PL} \cdots \succ_{PL} s_n$ and $r_1 \succ_{PL} r_2 \succ_{PL} \cdots \succ_{PL} r_m$ holds.

We define two properties, nonwastefulness and no justified envy, as in [8]. Wastefulness means that a student claims to move to an empty seat in a laboratory. Justified envy means that a student claims to exchange seats with another student whose rank in PL is lower than the student's.

Definition 5 *Matching* μ *is nonwasteful for domestic students if the following property is satisfied.* $\forall s \in S, c \in C, (c \succ_s \mu(s) \to (|\mu(c)| \geq q_c \vee |\mu(\mu(s))| \leq p_{\mu(s)}))$.

Matching μ *is nonwasteful for overseas students if the following property is satisfied.* $\forall r \in R, c \in C, (c \succ_r \mu(r) \to (|\mu(c)| \geq q_c \vee |\mu(\mu(r))| \leq p_{\mu(r)} \vee |\mu(c) \cap R| \geq qr_c \vee |\mu(\mu(r)) \cap R| \leq pr_{\mu(r)}))$.

Matching μ *is nonwasteful if it is nonwasteful for all students.*

Definition 6 *Matching* μ *has no justified envy between domestic students if the following property is satisfied.* $\forall s \in S, c \in C, (c \succ_s \mu(s) \to \forall s' \in \mu(c) \cap S(s' \succ_c s \vee s' \succ_{PL} s))$.

Matching μ *has no justified envy between overseas students if the following property is satisfied.* $\forall r \in R, c \in C, (c \succ_r \mu(r) \to \forall r' \in \mu(c) \cap R(r' \succ_c r \vee r' \succ_{PL} r))$.

Matching μ *has no justified envy from domestic students to overseas students if the following property is satisfied.* $\forall s \in S, c \in C, (c \succ_s \mu(s) \to \forall r \in \mu(c) \cap R(r \succ_c s \vee r \succ_{PL} s \vee |\mu(c) \cap R| \leq pr_c \vee qr_{\mu(s)} \leq |\mu(\mu(s)) \cap R|))$.

Matching μ has no justified envy from overseas students to domestic students if the following property is satisfied. $\forall r \in R, c \in C, (c \succ_r \mu(r) \rightarrow \forall s \in \mu(c) \cap S(s \succ_c r \vee s \succ_{PL} r \vee |\mu(\mu(r)) \cap R| \leq pr_{\mu(r)} \vee qr_c \leq |\mu(c) \cap R|))$.

Matching μ has no justified envy if μ does not have any justified envy shown above.

3 Multistage deferred acceptance algorithm for two types of students

The new multistage deferred acceptance algorithm for two types of students, two-type MSDA, is shown in Algorithm 1 and 2. The algorithm is based on the MSDA algorithm [8], which considers one type of student. The outline of the original MSDA algorithm is as follows. (1) Several students are reserved from the bottom of the PL list to fill the minimum quota of each laboratory. (2) The other students execute the standard deferred acceptance (DA) algorithm for the school in [1]. Since some students might fill the seats in the minimum quotas of some laboratories, the number of reserved students is decreased after a DA execution. Then, some number of students are no longer reserved. The students execute the DA algorithm. (3) These steps are repeated until the number of reserved students does not change. If the stable state is obtained, the number of currently reserved students equals the total number of empty seats to satisfy the minimum quota of each laboratory. Thus, the final DA is executed between the reserved students and empty seats for the minimum quotas. When there are two types of students, we need to change all the above steps.

Algorithm 1 (Subroutine) Modified DA algorithm for two types of students

Let p_c, q_c, pr_c, qr_c be current lower and upper quota of laboratory $c \in C$.
Each student t applies to his/her best laboratory by $\succ_t$. If t is rejected from a laboratory, t applies to the next laboratory by $\succ_t$. t repeats the procedure until t is not rejected.
For each laboratory c, let S_c and R_c be the set of domestic and overseas students currently applying to c, respectively. Let $T_c = S_c \cup R_c$
if $|R_c| > qr_c$ **then**
    Reject overseas students until $|R_c| = qr_c$ using the preference $\succ_c$.
else if $|S_c| > q_c - pr_c$ **then**
    Reject domestic students until $|S_c| = q_c - pr_c$ using the preference $\succ_c$.
else if $|R_c| \leq qr_c$, $|S_c| \leq q_c - pr_c$ and $|T_c| > q_c$ **then**
    Reject any student until $|T_c| = q_c$ using the preference $\succ_c$.
end if

The initialization of variables is executed at the top of Algorithm 2. The upper and lower quotas change during executions. The variables in the k-th iteration are written as q_c^k, p_c^k, and so on. First, some students are reserved to fill the minimum quotas for the laboratories. The set of reserved students V^k is the minimum number of students that satisfy the following four conditions.

1. $|V^k \cap R| \geq vr^k = \sum_{c \in C} pr_c^k$.
2. $|v^k \cap S| \geq vs^k = \sum_{c \in C} \max(0, p_c^k - qr_c^k)$.
3. $|V^k| \geq vt^k = \sum_{c \in C} p_c^k$.
4. If $t \in V^k$, any student t' who satisfies $t \succ_{PL} t'$ must satisfy $t' \in V^k$.

Algorithm 2 Two-type MSDA algorithm

Set $k = 0$, $V^0 = T$, $p_c^1 = p_c$, $q_c^1 = q_c$, $pr_c^1 = pr_c$, and $qr_c^1 = qr_c$ for all $c \in C$.
repeat
  $k = k + 1$
  Let $vs^k = \sum_{c \in C} \max(0, p_c^k - qr_c^k)$, $vr^k = \sum_{c \in C} pr_c^k$, and $vt^k = \sum_{c \in C} p_c^k$.
  Set V^k be the minimum set of students with the lowest priority according to $\succ_{PL}$ which satisfies $|V^k| \geq vt^k$, $|V^k \cap S| \geq vs^k$, and $|V^k \cap R| \geq vr^k$.
  if $V^{k-1} \setminus V^k \neq \emptyset$ **then**
    Execute modified DA mechanism on the students in $V^{k-1} \setminus V^k$. Let μ^k be the matching in this round.
    For each $c \in C$, set $q_c^{k+1} = q_c^k - |\mu^k(c)|$,
    $qr_c^{k+1} = \min(qr_c^k - |\mu^k(c) \cap R|, q_c^{k+1})$,
    $pr_c^{k+1} = \max(0, pr_c^k - |\mu^k(c) \cap R|)$, and
    $p_c^{k+1} = \max(0, p_c^k - |\mu^k(c) \cap S| - \max(pr_c^k, |\mu^k(c) \cap R|)) + pr_c^{k+1}$.
  else
    Let $C'(\subseteq C)$ be the set of laboratories which satisfies $p_c^k > 0$.
    if $|V^k \cap R| = vr^k$ **then**
      Execute DA algorithm on $V^k \cap R$ and every laboratory $c \in C'$ with $pr_c = qr_c = qr_c^k$. (that is, for the other laboratory $c' \notin C'$, set $pr_{c'} = qr_{c'} = 0$.)
      Execute MSDA algorithm on $V^k \cap S$ with $p_c = p_c^k - pr_c^k$ and $q_c = q_c^k - pr_c^k$ for laboratory $c \in C'$ and $p_{c'} = 0$ and $q_{c'} = q_{c'}^k$ for laboratory $c' \notin C'$.
      exit /* end of the algorithm */
    else if $|V^k \cap S| = vs^k$ **then**
      Execute DA algorithm on $V^k \cap S$ and every laboratory $c \in C'$ with $p_c = q_c = \max(p_c^k - qr_c^k, 0)$. (that is, for the other laboratory $c' \notin C'$, set $p_{c'} = q_{c'} = 0$.)
      Execute MSDA algorithm on $V^k \cap R$ with $pr_c = pr_c^k$ and $qr_c = qr_c^k$ for laboratory $c \in C'$ and $pr_{c'} = 0$ and $qr_{c'} = qr_{c'}^k$ for laboratory $c' \notin C'$
      exit /* end of the algorithm */
    else /* $|V^k \cap R| > vr^k$ and $|V^k| = vt^k$ */
      Execute two-type MSDA algorithm on the students in V^k and every laboratory $c \in C'$ with $pr_c = p_c = pr_c^k$, $qr_c = \min(qr_c^k, p_c^k)$, and $q_c = p_c^k$. (That is, for the other laboratory $c' \notin C'$, set $p_{c'} = q_{c'} = pr_{c'} = qr_{c'} = 0$.)
      exit /* end of the algorithm */
    end if
  end if
until forever

The set of students that satisfy (1), (2), and (3) is selected from the lowest student by $\succ_{PL}$. Because of the fourth condition, some extra students are selected in V^k. For example, consider the case when $vr^k = 1$, $vs^k = 2$, and $vt^k = 3$ and $(\dots, r_1, s_1, s_2, s_3)$ is the lower students in PL. In this case, the set of reserved students must be $V'_1 = \{r_1, s_1, s_2, s_3\}$ to achieve all the conditions. In the example, consider the case when s_1 is not reserved and the reserved students are $V''_1 = \{r_1, s_2, s_3\}$. Though V''_1 satisfies conditions (1)(2)(3), a PL-fair matching cannot be obtained. s_1 can apply to his/her favorite laboratory and s_1 is accepted to some laboratory c_1. In the next round, vt^{k+1}, vs^{k+1}, and vr^{k+1} are re-calculated and there can be a case when r_1 is no more included in V^{k+1}. Thus, r_1 can freely apply to his/her favorite laboratory c_1, but the seat is already taken by s_1. Since $r_1 \succ_{PL} s_1$, this can be a justified envy. To avoid this situation, a student t can be excluded from the reservation list only when every t' that satisfies $t' \succ_{PL} t$ is excluded.

After some students are reserved as V^k, all the other students $V^{k-1} \setminus V^k$ can freely apply to any laboratory. The algorithm is shown in Algorithm 1. As in the standard DA algorithm, student t applies to his/her laboratory according to $\succ_t$. The rejection rule for each laboratory must be changed because there are two types of students, and there are two maximum quotas for each laboratory. Let R_c and S_c be the current overseas and domestic students applying to c, respectively. Let $T_c = S_c \bigcup R_c$. If $|R_c| > qr_c$, the number of overseas students is more than the maximum quota. Thus, the number must be reduced to qr_c. Therefore, c rejects overseas students until $|R_c| = qr_c$ using $\succ_c$. If $|S_c| > q_c - pr_c$, the number of domestic students is greater than the allowed number, since at least pr_c overseas students must be accepted, and the total maximum quota is q_c. Thus, c rejects domestic students until $|S_c| = q_c - pr_c$ using $\succ_c$. Last, even if $|R_c| \leq qr_c$ and $|S_c| \leq q_c - pr_c$ are satisfied, the total number of applying students $|T_c|$ might satisfy $|T_c| > q_c$. In this case, c needs to reject some students until $|T_c| = q_c$. c can reject either domestic or overseas students since $|R_c| \leq qr_c$ and $|S_c| \leq q_c - pr_c$.

By the above assignment μ^k, some students might fill the seats for the minimum quotas. Thus, the number of reserved students might be reduced. Thus, the minimum and maximum quotas are recalculated.

For each laboratory c, the minimum quota of the overseas students is changed as $pr_c^{k+1} = \max(0, pr_c^k - |\mu^k(c) \cap R|)$, since $|\mu^k(c) \cap R|$ overseas students are accepted.

The minimum quota of all students is calculated as follows.

(Case 1) When overseas students are accepted more than the minimum quota pr_c^k, that is, $|\mu^k(c) \cap R| \geq pr_C^k$, the remaining minimum quota that must be filled is $\max(0, p_c^k - |\mu^k(c) \cap R| - |\mu^k(c) \cap S|)$. Note that in this case, $pr_c^{k+1} = 0$.

(Case 2) When overseas students are accepted less than the minimum quota pr_c^k, that is, $|\mu^k(c) \cap R| < pr_c^k$, the remaining minimum quota is the sum of the remaining minimum quota for the overseas students pr_c^{k+1} and the remaining minimum quota for both students $\max(0, p_c^k - pr_c^k - |\mu^k(c) \cap S|)$. These two cases are summarized in one equation $p_c^{k+1} = \max(0, p_c^k - \max(pr_c^k, |\mu^k(c) \cap R|) - |\mu^k(c) \cap S|) + pr_c^{k+1}$.

Next, the maximum quota of all students is $q_c^{k+1} = q_c^k - |\mu^k(c)|$. The maximum quota of overseas students is $qr_c^{k+1} = \min(qr_c^k - |\mu^k(c) \cap R|, q_c^{k+1})$, since the maximum quota of overseas students cannot be more than the maximum quota of all students.

After the new maximum and minimum quotas are updated, the reversed students are selected using the new quotas. Then, the newly released students execute the modified DA algorithm. This procedure is repeated until there is no change in the set of reserved students.

If there is no change in the reserved student set, that is, $V^{k-1} = V^k$, then we need to assign the remaining V^k. There are three cases for V^k.

(Case 1) $|V^k \cap R| = vr^k$. The number of reserved overseas students is the minimum, thus, they must be assigned to the empty slots for the overseas students. The assignment can be executed by a standard DA algorithm with the maximum quotas. The remaining domestic students in V^k can be assigned using the MSDA algorithm with the maximum and minimum quotas.

(Case 2) $|V^k \cap S| = vs^k$. The number of reserved domestic students is the minimum. The assignment can be executed by a standard DA algorithm with the maximum quotas. The remaining students can be assigned using the standard MSDA algorithm with the minimum and maximum quotas.

(Case 3) The remaining case is $|V^k \cap R| > vr^k$, $|V^k \cap S| > vs^k$, and $|V^k| = vt^k$.

In this case, we can execute the two-type MSDA again for the reserved set of students with the new quotas. Let $C' = c \in C | p_c^k > 0$. About the overseas students, $pr_c = pr_c^k$ and $qr_c = \min(qr_c^k, p_c^k)$ for every laboratory $c \in C'$. The maximum quota of overseas students is changed because the laboratory must accept no more than the minimum quota p_c^k. The minimum and the maximum quotas of all students can be changed as follows. $p_c = pr_c^k$ and $q_c = p_c^k$ for every laboratory $c \in C'$. This condition means there is no minimum quota restriction for either student. Since the number of empty seats in C' equals the number of students, satisfying the maximum quotas automatically satisfies the minimum quota conditions of both students. The students must fill all the slots in C', thus $p_{c''} = q_{c''} = pr_{c''} = qr_{c''} = 0$ for every laboratory $c'' \notin C'$.

The recursive procedure always terminates since the number of reserved overseas students decreases.

Theorem 1. *The two-type MSDA algorithm is strategyproof, nonwasteful, and PL fair.*

The proof is omitted because of the page limitation.

4 Conclusion

We showed the two-type MSDA algorithm which satisfies strategyproof, nonwasteful, and PL-fair. It seems very hard to generalize this algorithm to more than two types of students, because if there are only two types of students, a seat that cannot be filled by a type of student must be filled by the other type of student. This fact makes the algorithm simple.

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