A three-player envy-free division protocol for mixed manna

Yuki Okano and Yoshifumi Manabe
Faculty of Informatics, Kogakuin University
Shinjuku, Tokyo Japan
manabe@cc.kogakuin.ac.jp

Abstract- *This paper proposes a three player envy-free assignment protocol of a divisible good whose utility is positive for some players and negative for the others. Such a good is called mixed manna. For mixed manna, current discrete envy-free cake-cutting or chore-division protocols cannot be applied. A naive protocol to achieve an envy-free division of mixed manna for three players needs an initial division of given mixed manna into 8 pieces. This paper shows a new three player envy-free division protocol which needs an initial division into two pieces. After the initial division, it is shown that each of the pieces can be divided using modified current envy-free cake-cutting and chore-division protocols.*

I. INTRODUCTION

This paper proposes a three player envy-free assignment protocol of a divisible good whose utility is positive for some players and negative for the others. Many works have been done for the cake-cutting problem, where a divisible good has some positive utility to every player. There are some surveys to these problems [5, 6, 11, 12]. Some number of works have been done for the chore division problem, where a divisible good has some negative utility to every player [7, 8, 9]. The problem can be used to assign dirty work among people. There are some cases when a portion of a divisible good has some positive utility to some players but the same portion has some negative utility to the other players. For example, a child does not like chocolate but another child likes chocolate on a cake. Nations do not want a region where people who believe in different religions live. A good which has such a property is called mixed manna. Very few works have been done for fair division of divisible mixed manna [13].

There are several assignment results for given number of indivisible mixed manna [3, 4]. Ref. [13] proved the existence of a connected envy-free division of divisible mixed manna by three players. However, finding such a division cannot be done by a finite number of queries. Thus, a simple protocol to divide divisible mixed manna is necessary. The most widely discussed property that fair division protocols must satisfy is envy-freeness [5, 12]. An envy-free cake division among any number of players can be done by a fixed number discrete operations [2]. An envy-free chore division among any number of players can also be done by a fixed number of discrete operations [7]. This paper discusses envy-free division of a mixed manna. The above cake-cutting or chore-division protocols cannot be used to divide a mixed manna. A naive envy-free division protocol is shown in [13], which works for any number of player, needs many initial divisions. When the number of players is 3, the manna must be initially divided into 8 pieces. Thus the protocol is not efficient. This paper proposes a new envy-free mixed manna division protocol for three players in which the number of initial cuts is one. After the initial division, it is shown that each of the pieces can be divided using modified current envy-free cake-cutting and chore-division protocols.

II. PRELIMINARIES

Throughout the paper, a mixed manna is a heterogeneous good that is represented by interval [0, 1] on areal line. It can be cut anywhere between 0 and 1. Each player P_i has a utility function, μ_i, that has the following properties.

(1) $\mu_i(X)$ can be positive or negative for any $X \subseteq [0, 1]$.

(2) For any X_1 and X_2 such that $X_1 \cap X_2 = \emptyset$, $\mu_i(X_1 \cup X_2) = \mu_i(X_1) + \mu_i(X_2)$.

Note that $\mu_i(X)$ and $\mu_j(X)$ $(i \neq j)$ are independent, thus $\mu_i(X) > 0$ and $\mu_j(X) < 0$ for some X might occur.

The tuple of the utility function of $P_i (i = 1,2, \dots, n)$ is denoted as $(\mu_1, \mu_2, \dots, \mu_n)$. No player has knowledge of the utility functions of the other players.

An n-player division protocol, f, assigns some portions of [0, 1] to each player such that every portion of [0, 1] is assigned to some player. This means that no portion of the manna is discarded. We denote $f_i(\mu_1, \mu_2, \dots, \mu_n)$ as the set of portions assigned to player P_i by f, when the tuple of the utility function is $(\mu_1, \mu_2, \dots, \mu_n)$.

All players are risk-averse, namely they avoid gambling. They try to maximize the worst case utility they can obtain.

Several desirable properties of fair division protocols have been defined [12]. One of the most widely considered property is envy-freeness. The definition of envy-free is as follows: for any $i, j (i \neq j)$, $\mu_i\left(f_i(\mu_1, \mu_2, \dots, \mu_n)\right) \geq \mu_i\left(f_j(\mu_1, \mu_2, \dots, \mu_n)\right)$. Envy-free means that every player thinks he has obtained more than or equal value to any other player.

III. A NAÏVE PROTOCOL FOR MIXED MANNA

First, let us review an easy example of two player case shown in [13]. The divide-and-chose protocol for the cake-cutting problem by two players works for a mixed manna. The divide-and-choose is as follows: the first player, called Divider, cuts the cake into two pieces. The other player, called Chooser, selects the piece he wants among the two pieces. Divider obtains the remaining piece. The reason that divide-and-choose works for a mixed manna is as follows. Since Divider is a risk-adverse player, Divider cuts the manna into two pieces $[0, x]$ and $[x, 1]$, such that $\mu([0, x]) = \mu([x, 1]) = 1/2\mu([0, 1])$ for Divider, whenever $\mu([0, 1]) \geq 0$ or $\mu([0, 1]) < 0$ holds. Otherwise, Chooser might select the better piece and Divider might obtain the worse piece. Since Divider cuts the manna into two equal utility pieces, Divider does not envy Chooser. Chooser selects the better piece among the two pieces. Thus Chooser does not envy Divider. Therefore, Divide-and-choose can be used for an envy-free division of mixed manna.

Next, let us consider a three player case. Selfridge-Conway protocol [12], shown in Fig. 1, is a discrete cake-cutting protocol to achieve envy-freeness. This protocol cannot be used for mixed manna by several reasons. Though P_1 can cut the manna into three pieces X_1, X_2, and X_3 whose utilities are the same for P_1, there can be a case when $\mu_2(X_1) > 0$ and $\mu_2(X_2) < 0$. In this case, P_2 cannot cut L from X_1 so that $\mu_2(X_1 - L) = \mu_2(X_2)$. Even if $\mu_2(X_1) > 0$, $\mu_2(X_2) > 0$, and P_2 can cut L from X_1, there can be a case when $\mu_1(L) < 0$ and $X_1' = X_1 - L$ becomes the best piece for P_1. If P_2 or P_3 selects X'_1, P_1 envies the player. A similar situation occurs at the assignment of L. Therefore, the Selfridge-Conway protocol cannot be used for mixed manna. Three player envy-free chore division protocol shown in [10] cannot be used for mixed manna by a similar reason.

A naive envy-free assignment protocol for a mixed manna is shown in [13]. First, divide the manna as follows:

X_{123} such that any portion $x \subseteq X_{123}$ satisfies $\mu_i(x) \geq 0$ for every player $P_i (i = 1,2,3)$.

Begin
    P_1 cuts into three pieces so that the utilities of the pieces is the same for P_1.
    Let X_1, X_2, X_3 be the pieces where $\mu_2(X_1) \geq \mu_2(X_2) \geq \mu_2(X_3)$.
    If $\mu_2(X_1) > \mu_2(X_2)$ then
        P_2 cuts L from X_1 so that $\mu_2(X_1') = \mu_2(X_2)$ where $X_1' = X_1 - L$.
    P_3 selects the largest (for P_3) among X'_1, X_2, and X_3.
    If X_1' remains then
        P_2 must select X'_1.
        Let (P_a, P_b) be (P_3, P_2).
    Else
        P_2 selects X_2 (the largest for P_2).
        Let (P_a, P_b) be (P_2, P_3).
    P_1 obtains the remaining piece.
    If L is not empty then
        P_a cuts L into three pieces (so that P_a considers their utilities are the same).
        P_b, P_1, and P_a selects one piece in this order.
End.

Figure 1. Selfridge-Conway three-player envy-free cake-cutting protocol [12].

$X_{ij}(i,j = 1,2,3, i < j)$ such that any portion $x \subseteq X_{ij}$ satisfies $\mu_i(x) \geq 0, \mu_j(x) \geq 0$, and $\mu_k(x) < 0$ for the other player P_k.

$X_i(i = 1,2,3)$ such that any portion $x \subseteq X_i$ satisfies $\mu_i(x) \geq 0$ and $\mu_j(x) < 0$ for $j \neq i$.

The remaining portion X_4 such that any portion $x \subseteq X_4$ satisfies $\mu_i(x) < 0$ for $i = 1,2,3$.

Then, the Selfridge-Conway protocol is executed among all players for X_{123}. Divide-and-choose is executed to X_{ij} between P_i and P_j. X_i is given to P_i. Last, three-player envy-free chore division protocol [10] is executed for X_4. Though this procedure achieves an envy-free assignment, the procedure to initially divide the manna is complicated. The mixed manna must be divided into the above 8 pieces. Note that each of the 8 pieces might not be connected. For example, disconnected multiple portions might satisfy $\mu_i(x) \geq 0$ for all players, thus X_{123} might consist of multiple portions. Thus, the number of cuts to obtain the above 8 pieces is not bounded. When $P_i(i = 1,2,3)$ needs to cut the manna c_i times to divide into non-negative regions and negative regions for P_i, the manna needs to be cut $c_1 + c_2 + c_3$ times in the worst case. This paper considers reducing the procedure of the initial division.

IV. A NEW PROTOCOL FOR MIXED MANNA

This section shows a new three-player envy-free division protocol for mixed manna in which the number of the initial division is reduced. Initially, cut the manna as follows:

X^+ such that any portion of $x \subseteq X^+$ satisfies $\mu_1(x) \geq 0$.

X^- such that any portion of $x \subseteq X^-$ satisfies $\mu_1(x) < 0$.

$X^+(X^-)$ is the portion with non-negative (negative) utility for P_1. The manna must be cut c_1 times. Note that c_1 can be selected as $\min_i c_i$. Thus the number of cuts necessary for the initial division is reduced at least 1/3. $X^+(X^-)$ might consist of multiple disconnected pieces. In the case, the disconnected pieces are collected to make one piece. X^+ and X^- might contain both positive and negative portions for the other players.

First, we show an envy-free assignment of X^+ in Fig. 2, in which the Selfridge-Conway protocol is slightly modified. Initially, P_1 cuts X^+ into three pieces. If both of P_2 and P_3 think at most one piece has a non-negative utility, an envy-free assignment is easily obtained. If P_2 or P_3 thinks that at least two pieces have a non-negative utility, the Selfridge-Conway protocol can be executed because P_1 thinks any portion of X^+ has a non-negative utility.

[Theorem 1] The assignment result of X^+ by the protocol in Fig. 2 is envy-free.

(Proof) First, consider the case when both of P_2 and P_3 consider that at most one piece among X_1^+, X_2^+, and X_3^+ has a non-negative utility. Consider the subcase when both of P_2 and P_3 think the same piece, say X_1^+, has a non-negative utility. P_2 and P_3 execute

Begin
  P_1 cuts into three pieces X_1^+, X_2^+, and X_3^+ so that $\mu_1(X_1^+) = \mu_1(X_2^+) = \mu_1(X_3^+)$.
  If P_2 and P_3 consider at most one piece has a non-negative utility then
    If P_2 and P_3 consider the same piece (say, X_1^+) has a non-negative utility then
      P_2 and P_3 execute Divide-and-choose on X_1^+.
      P_1 obtains X_2^+ and X_3^+.
    Else
      Each of P_2 and P_3 obtains at most one piece with a non-negative utility.
      P_1 obtains the remaining piece(s).
  Else
    Let P_2 be a player who considers two pieces have some non-negative utility.
    Rename the pieces so that $\mu_2(X_1^+) \geq \mu_2(X_2^+) \geq \mu_2(X_3^+)$.
    Execute Selfridge-Conway protocol from step 3 with the three pieces.
End

Figure 2: Three-player envy-free protocol for X^+.

Divide-and-choose on X_1^+. Let P_2 and P_3 obtain X_{12}^+ and X_{13}^+, respectively. Since $X_1^+ = X_{12}^+ \cup X_{13}^+$ and any portion of X_1^+ has a non-negative utility for P_1, $\mu_1(X_{12}^+) \leq \mu_1(X_1^+) = \mu_1(X_2^+)$ and $\mu_1(X_{13}^+) \leq \mu_1(X_1^+) = \mu_1(X_2^+)$ hold. Since P_1 obtains X_2^+ and X_3^+, P_1 does not envy P_2 or P_3. P_2 and P_3 do not envy each other because of the envy-freeness of Divide-and-choose. P_2 does not envy P_1, since $\mu_2(X_2^+) < 0$ and $\mu_2(X_3^+) < 0$ hold. Similarly, P_3 does not envy P_1.

Next consider the subcase when there is no piece that has a non-negative utility for both of P_2 and P_3. In this case, P_2 and P_3 can obtain at most one piece whose utility is not negative for the player. P_1 obtains the remaining pieces, which have a negative utility for both of P_2 and P_3. Thus, every player does not envy the other players.

Next, consider the case when one player, say P_2, thinks two pieces have a non-negative utility. In this case, the Selfridge-Conway protocol can be executed. The reason is as follows. P_2 can cut L from X_1^+ if $\mu_2(X_1^+) > \mu_2(X_2^+)$ since both of these utilities are non-negative. Each player can select one piece among $X_1'^+$, X_2, and X_3. The assignment result is envy-free, since P_3 selects first, there are two equal utility pieces for P_2, and P_1 can obtain one full-size piece (Note that any portion of X^+ has non-negative utility for P_1, thus $\mu_1(X_1'^+) \leq \mu_1(X_1^+)$ holds). Envy-free assignment of L can also be realized. Even if the utility is positive or negative, P_a can cut L into three pieces with the same utility. P_a does not envy any other players since the three pieces have the same utility. P_b does not envy any other players since P_b selects first. P_1 does not envy P_b since P_b does not obtain 1/3 of X^+ (Note again P_1 thinks any portion of X^+ has a non-negative utility). P_1 does not envy P_a since P_1 selects before P_a. □

Next, X^- needs to be assigned. We use the three player envy-free chore division shown in [10]. Since the protocol uses Austin's moving knife procedure [1], the protocol is not discrete. Discrete envy-free chore division protocol in [7] or [12] cannot be used by a similar reason why Selfridge-Conway cannot be used for mixed manna. The protocol in [10] is shown in Fig. 3. Let Y be the chore to be divided. Note that the protocol assumes that any portion of chore has negative utility for any players.

This protocol cannot be used to divide X^- as it is because the players cannot execute Austin's moving knife protocol. Austin's protocol by two player P_1 and P_2 for a cake X with positive utility is shown in Fig. 4. If both of P_1 and P_2 are honest, $\mu_i(X_1 \cup X_3) = \mu_i(X_2) = 1/2\mu_i(X)(i = 1,2)$, thus both players obtain half of the cake.

Austin's protocol cannot be used for mixed manna Y. Let us assume that Y_2 is mixed manna for P_3. If P_3 has two knives, P_3 cannot move the two knives while keeping the utility of the portion between the two knifes is half. Suppose that $\mu_3(Y_2) = 10$. P_3 initially sets the left knife at the left end and the right knife at the position so that the utility of the portion between the two knives is 5. Consider the case at that position, the portion just right of the left end has a negative utility and the portion just right of the right knife has a positive utility for P_3. In this situation, moving the right knife to right increases the total utility between the two knives. Moving the left knife to right also increases the total utility. Thus, it is impossible for P_3 to move the two knives while keeping the utility the same

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Begin
  Consider Y as a cake, that is, set each player's utility function μ'_i = −μ_i (i = 1,2,3).
  Execute a three player envy-free cake division protocol using μ'_i.
  Let Y_i (i = 1,2,3) be the portion P_i obtained. (Note that P_i thinks Y_i is the worst among three portions.)
  For i = 1 to 3 Do
    P_i divides Y_i into two pieces Y_{i,0} and Y_{i,1} so that μ_i(Y_{i,0}) = μ_i(Y_{i,1}) = 1/2μ_i(Y_i).
    These two pieces are assigned to the other players so that P_j (j ≠ i) has gotten no worse than 1/2μ_j(Y_i).
      (The above assignment can be achieved by the following procedure:
       P_i and another player, say P_j, executes Austin's moving knife procedure on Y_i.
       Then μ_i(Y_{i,0}) = μ_i(Y_{i,1}) = 1/2μ_i(Y_i) and μ_j(Y_{i,0}) = μ_j(Y_{i,1}) = 1/2μ_j(Y_i) are achieved.
       The other player, P_k, first selects one piece, Y_{i,α} (α = 0 or 1).
       Then, μ_k(Y_{i,α}) ≥ μ_k(Y_{i,1−α}), therefore μ_k(Y_{i,α}) ≥ 1/2μ_k(Y_i) )
End.
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Figure 3: Three-player envy-free chore division protocol [10].

value (5). The above problem can be solved by modifying the protocol. At step 9, P_1 always plays the role of having two knives for every division of $Y_i (i = 1,2,3)$. The modified protocol is shown in Fig. 5.

[Theorem 2] The assignment result of X^- by the protocol in Fig. 5 is envy-free.

(Proof) Since the utility of any portion of X^- is negative for P_1, it can be divided to achieve envy-free when -1 is multiplied to the utility functions using Theorem 1. Suppose that P_i obtains $X_i^- (i = 1,2,3)$. By the original utility, $\mu_i(X_i^-) \leq \mu_i(X_j^-)(i = 1,2,3, j \neq i)$ holds. For X_1^- and X_2^-, P_1 and P_2 execute Austin's protocol. P_1 and P_3 execute Austin's protocol for X_3^-. If P_1 has the two knives in these executions, it is possible for P_1 to move the two knives while keeping the utility of the portion between the knives is always $1/2\mu_1(X_i^-)$, because any portion of X_i^- has a negative utility for P_1. While P_1 moves the two knives, the other player (P_2 or P_3) can say stop when the utility of the portion between the knives becomes half for the player, even if the piece is mixed manna for the player. The reason is as follows. When P_1 initially sets the left knife at the left end of the piece, let the right knife is at some point p. At the end of moving the knives, the left knife comes to p and the right knife comes to the right end of the piece. Without loss of the generality, suppose that at the beginning P_2 thinks the utility of the portion between the two knives is more than half of the whole piece. In the case, the utility of the portion from p to the right end must be less than half. Therefore, during moving the knives, there must be at least one point when the utility of the portion between the two knives becomes the half of the utility for P_2. The execution between P_1 and P_3 is similar to the case of P_2.

Thus, the utility of the divided piece satisfies $\mu_1(X_{i,0}^-) = \mu_1(X_{i,1}^-) = 1/2\mu_1(X_i^-)$, $\mu_2(X_{i,0}^-) = \mu_2(X_{i,1}^-) = 1/2\mu_2(X_i^-)(i = 1,2)$, and $\mu_1(X_{3,0}^-) = \mu_1(X_{3,1}^-) = 1/2\mu_1(X_3^-)$, and $\mu_3(X_{3,0}^-) = \mu_3(X_{3,1}^-) = 1/2\mu_3(X_3^-)$.

Let $s_i \in \{0,1\}(i = 1,2,3)$ be the index of the piece selected by $P_3(i = 1,2)$ and $P_2(i = 3)$. Since every player selects the better piece, $\mu_3(X_{i,s_i}^-) \geq \mu_3(X_{i,1-s_i}^-)(i = 1,2)$ and $\mu_2(X_{3,s_3}^-) \geq \mu_2(X_{3,1-s_3}^-)$ are satisfied. Therefore, $\mu_3(X_{i,s_i}^-) \geq 1/2\mu_3(X_i^-)(i = 1,2)$ and $\mu_2(X_{3,s_3}^-) \geq 1/2\mu_2(X_3^-)$ are satisfied.

1: Begin
2: P_1 has two knives. P_1 sets the left knife at the left end of the cake X.
3: P_1 sets the right knife at a position that satisfies the utility of the portion of the cake between the two knives is $1/2\mu_1(X)$.
4: If the utility of the portion is $1/2\mu_2(X)$
5: P_2 calls `stop'.
6: Else
7: P_1 moves the two knives simultaneously to right so that the utility of the portion between the two knives is $1/2\mu_1(X)$.
8: During the move, P_2 calls `stop' if the utility of the portion between two knives is $1/2\mu_2(X)$.
9: When `stop' is called, P_1 stops moving the knives and cuts the cake at the positions of the two knives.
10: The cake is cut into at most three pieces, X_1, X_2 and, X_3(let X_2 be the portion between the two knives).
11: P_1 and P_2 execute a coin-toss to decide which player obtains the pair (X_1, X_3) or X_2.
12: End.

Figure 4: Austin's moving knife protocol [1].

1: Begin
2: Set each player's utility function $\mu_i' = -\mu_i(i = 1,2,3)$.
3: Execute the three player envy-free division protocol in Fig. 2 on X^- using μ_i'.
4: Let $X_i^-(i = 1,2,3)$ be the portion P_i obtained. (Note that P_i thinks X_i^- is the worst among three portions.)
5: For $i = 1$ to 2 Do
6: P_1 and P_2 execute Austin's protocol on X_i^-, where P_1 has two knives. Suppose that X_i^- is divided to $X_{i,0}^-$ and $X_{i,1}^-$.
7: P_3 selects $X_{i,0}^-$ or $X_{i,1}^-$. The remaining piece is given to P_{3-i}.
8: P_1 and P_3 execute Austin's protocol on X_3^-, where P_1 has two knives. Suppose that X_3^- is divided to $X_{3,0}^-$ and $X_{3,1}^-$.
9: P_2 selects $X_{3,0}^-$ or $X_{3,1}^-$. The remaining piece is given to P_1.
10: End.

Figure 5: Three-player envy-free division protocol for X^-.

P_1 obtains $Z_1 = X^-_{2,1-s_2} \cup X^-_{3,1-s_3}$. P_2 obtains $Z_2 = X^-_{1,1-s_1} \cup X^-_{3,s_3}$. P_3 obtains $Z_3 = X^-_{1,s_1} \cup X^-_{2,s_2}$. The utilities of these pieces satisfies the following inequalities.

$\mu_1(X^-_{3,1-s_3}) = \mu_1(X^-_{3,s_3})$ and $\mu_1(X^-_{2,1-s_2}) = 1/2\mu_1(X^-_2) \geq 1/2\mu_1(X^-_1) = \mu_1(X^-_{1,1-s_1})$. Thus $\mu_1(Z_1) \geq \mu_1(Z_2)$.

$\mu_1(X^-_{2,1-s_2}) = \mu_1(X^-_{2,s_2})$ and $\mu_1(X^-_{3,1-s_3}) = 1/2\mu_1(X^-_3) \geq 1/2\mu_1(X^-_1) = \mu_1(X^-_{1,s_1})$. Thus $\mu_1(Z_1) \geq \mu_1(Z_3)$.

$\mu_2(X^-_{3,s_3}) = \mu_2(X^-_{3,1-s_3})$ and $\mu_2(X^-_{1,1-s_1}) = 1/2\mu_2(X^-_1) \geq 1/2\mu_2(X^-_2) = \mu_2(X^-_{2,1-s_2})$. Thus $\mu_2(Z_2) \geq \mu_2(Z_1)$.

$\mu_2(X^-_{1,1-s_1}) = \mu_2(X^-_{1,s_1})$ and $\mu_2(X^-_{3,s_3}) \geq 1/2\mu_2(X^-_3) \geq 1/2\mu_2(X^-_2) = \mu_2(X^-_{2,s_2})$. Thus $\mu_2(Z_2) \geq \mu_2(Z_3)$.

$\mu_3(X^-_{2,s_2}) \geq \mu_3(X^-_{2,1-s_2})$ and $\mu_3(X^-_{1,s_1}) = 1/2\mu_3(X^-_1) \geq 1/2\mu_3(X^-_3) = \mu_3(X^-_{3,1-s_3})$. Thus $\mu_3(Z_3) \geq \mu_3(Z_1)$.

$\mu_3(X^-_{1,s_1}) \geq \mu_3(X^-_{1,1-s_1})$ and $\mu_3(X^-_{2,s_2}) = 1/2\mu_3(X^-_2) \geq 1/2\mu_3(X^-_3) = \mu_3(X^-_{3,s_3})$. Thus $\mu_3(Z_3) \geq \mu_3(Z_2)$.

Therefore, envy-freeness is satisfied. □

V. CONCLUSION

This paper showed a three-player envy-free division protocol for mixed manna. This protocol reduces the initial division by the naïve protocol. Note that the initial division might need many cuts, thus the number of cuts is not bounded. In addition, moving knife protocols are not efficient. The most important open problem is obtaining a discrete protocol. In addition, each player's role in the protocol differs among the players and meta-envy [14] exists. A meta-envy-free protocol is necessary for the ideal fairness.

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