# A three-player envy-free discrete division protocol for mixed manna

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**Abstract.** This paper proposes a three-player envy-free discrete assignment protocol of a divisible good, in which the utility of some portion of the good can be positive for some players and negative for the others. Such a good is called mixed manna. For mixed manna, current discrete envy-free cake-cutting or chore-division protocols cannot be applied. A naive protocol to achieve an envy-free division of mixed manna for three players needs an initial division of given mixed manna into eight pieces. This paper shows a new three-player envy-free discrete division protocol that needs an initial division into two pieces. After the initial division, it is shown that each of the pieces can be divided by modifying current envy-free cake-cutting and chore-division protocols.

**Keywords:** cake-cutting · mixed manna · chore-division · divisible good · envy-free.

## 1 Introduction

This paper proposes a three-player envy-free assignment protocol of a divisible good in which the utility of some portion of the good can be positive for some players and negative for the others. Many works have been done for the cake-cutting problem, where a divisible good has some positive utility to every player. There are some surveys to these problems [7,8,14,17,18]. Some number of works have been done for a chore division problem, where a divisible good has some negative utility to every player [9,10,12,16]. The problem can be used to assign dirty work among people. There are some cases when a portion of a divisible good has some positive utility to some players but the same portion has some negative utility to the other players. For example, a child does not like chocolate but another child likes chocolate on a cake. A nation does not want a region where the religion believed by the residents is different from the national religion. A good which has such a property is called mixed manna. Very few works have been done for fair divisions of divisible mixed manna [19].

There are several assignment results for a given number of indivisible mixed manna [1–3, 5, 6, 11]. Ref. [19] proved the existence of a connected envy-free division of divisible mixed manna by three players. However, finding such a division cannot be done by a finite number of queries. Thus, a simple protocol to divide divisible mixed manna is necessary. The most widely discussed property that fair division protocols must satisfy is envy-freeness [7, 18]. An envy-free cake division among any number of players can be done by a fixed number of discrete operations [4]. An envy-free chore division among any number of players can also be done by a fixed number of discrete operations [9]. This paper discusses an envy-free division of mixed manna. The above cake-cutting or chore-division protocols cannot be used to divide mixed manna. A naive envy-free division protocol is shown in [19], which works for any number of players, needs many initial divisions. When the number of players is three, the manna must be initially divided into eight pieces. Thus, the protocol is not efficient. The protocol in [15] for three players needs an initial division into two pieces. Though the protocol achieves envy-free, the protocol is not discrete sine it uses a moving-knife procedure. We show a new discrete protocol for three players in which the initial division is the same as the one in [15]. After the initial division, it is shown that each of the pieces can be divided by modifying current envy-free cake-cutting and chore-division protocols.

Section 2 defines the problem. Section 3 shows the naive protocol. Section 4 shows the new protocol. Section 5 concludes the paper.

## 2 Preliminaries

Throughout the paper, mixed manna is a heterogeneous good that is represented by the interval $[0, 1]$ on a real line. It can be cut anywhere between 0 and 1. Each player $P_i$ has a utility function, $\mu_i$, which has the following properties.

1. $\mu_i(X)$ can be positive or negative for any $X = [a, b] (0 \geq a < b \leq 1)$.
2. For any $X_1$ and $X_2$ such that $X_1 \cap X_2 = \emptyset$, $\mu_i(X_1 \cup X_2) = \mu_i(X_1) + \mu_i(X_2)$.

Note that $\mu_i(X)$ and $\mu_j(X) (i \neq j)$ are independent, thus $\mu_i(X) > 0$ and $\mu_j(X) < 0$ for some $X$ might occur.

Note that if the first condition is changed as $\mu_i([a, b]) \geq 0$ for any $a, b$, and $i$, the problem becomes cake-cutting. If the condition is changed as $\mu_i([a, b]) \leq 0$ for any $a, b$, and $i$, the problem becomes chore-division.

The tuple of the utility function of $P_i (i = 1, 2, \ldots, n)$ is denoted as $(\mu_1, \mu_2, \ldots, \mu_n)$. No player knows the utility functions of the other players.

An $n$-player division protocol, $f$, assigns some portions of [0, 1] to each player such that every portion of [0, 1] is assigned to some player. This means that no portion of the manna is discarded. We denote $f_i(\mu_1, \mu_2, \ldots, \mu_n)$ as the set of portions assigned to the player $P_i$ by $f$ when the tuple of the utility functions is $(\mu_1, \mu_2, \ldots, \mu_n)$.

All players are risk-averse, namely, they avoid gambling. They try to maximize the worst utility they might obtain.

Several desirable properties of fair division protocols have been defined [7, 18]. One of the most widely considered property is envy-freeness. The definition of envy-free is as follows: for any $i, j (i \neq j), \mu_i(f_i(\mu_1, \mu_2, \ldots, \mu_n)) \geq \mu_i(f_j(\mu_1, \mu_2, \ldots, \mu_n))$. Envy-free means that every player thinks he has obtained more than or equal value to any other player.

## 3 A naive protocol for mixed manna

First, let us review an easy example of the two-player case shown in [19]. The Divide-and-chose protocol for the cake-cutting problem by two players works for any mixed manna. The Divide-and-choose is as follows: the first player, called Divider, cuts the cake into two pieces. The other player, called Chooser, selects the piece he wants among the two pieces. Divider obtains the remaining piece. The reason that Divide-and-choose works for mixed manna is as follows. Since Divider is a risk-averse player, Divider cuts the manna into two pieces $[0, x]$ and $[x, 1]$, such that $\mu_D([0, x]) = \mu_D([x, 1]) = 1/2\mu_D([0, 1])$ for Divider, whenever $\mu_D([0, 1]) \geq 0$ or $\mu_D([0, 1]) < 0$ holds. Otherwise, Chooser might select the better piece and Divider might obtain the worse piece. Since Divider cuts the manna into two equal utility pieces, Divider does not envy Chooser. Chooser selects the better piece among the two pieces. Thus, Chooser does not envy Divider. Therefore, Divide-and-choose can be used for an envy-free division of any mixed manna.

Next, let us consider a three-player case. Selfridge-Conway protocol [18], shown in Fig. 1, is a discrete cake-cutting protocol to achieve envy-freeness. The outline of the protocol is as follows. First, $P_1$ cuts the cake into three pieces whose utilities are the same for $P_1$. If $P_2$ thinks the utility of the largest piece, $\mu_2(X_1)$, is larger than the one of the second-best piece, $\mu_2(X_2)$, $P_2$ cuts $L$ from $X_1$ so that $\mu_2(X_1 - L) = \mu_2(X_2)$. If $P_2$ thinks $\mu_2(X_1) = \mu_2(X_2)$, $P_2$ does nothing. Then, $P_3$ selects the best piece among $X_1 - L$, $X_2$, and $X_3$. Next, $P_2$ selects one piece between the remaining two pieces. In this case, if $X_1 - L$ remains, $P_2$ must select the piece. $P_1$ obtains the remaining piece. Note that this assignment is envy-free. Since $P_3$ first selects, $P_3$ obtains the best piece for $P_3$. Since $P_2$ makes two equal-value pieces, $P_2$ can obtain one of the best pieces whatever $P_3$ selects. Though $P_1$ obtains the remaining piece, the piece is not cut by $P_2$, thus it is one of the best pieces for $P_1$.

Next, $L$ needs to be assigned if $L$ is cut from $X_1$ at Step 5. Let $P_b$ be the player who obtained $X_1 - L$ between $P_2$ and $P_3$. Let $P_a$ be the other player. $P_a$ cuts $L$ into three pieces whose utilities are the same for $P_a$. Then $P_b$, $P_1$, and $P_a$ select one piece in this order. $P_b$ does not envy the other players since $P_b$ selects first. $P_a$ does not envy the other players because the utilities of the three pieces are the same. $P_1$ does not envy $P_a$ because $P_1$ selects earlier than $P_a$. The reason why $P_1$ does not envy $P_b$ is as follows. Since $P_b$ obtains $X_1 - L$, the total utility of $P_b$ is less than the utility of $X_1$ for $P_1$. $P_1$ obtains at least $\mu_1(X_1)$, thus $P_1$ does not envy $P_b$.

This protocol cannot be used for mixed manna for several reasons. Though $P_1$ can cut the manna into three pieces $X_1$, $X_2$, and $X_3$ whose utilities are the same for $P_1$, there can be a case when $\mu_2(X_1) > 0$ and $\mu_2(X_2) < 0$. In this case, $P_2$ might not be able to cut $L$ from $X_1$ so that $\mu_2(X_1 - L) = \mu_2(X_2)$. Even if $\mu_2(X_1) > 0$, $\mu_2(X_2) > 0$, and $P_2$ can cut $L$ from $X_1$, there can be a case when $\mu_1(L) < 0$ and $X_1' = X_1 - L$ becomes the best piece for $P_1$. If $P_2$ or $P_3$ selects $X_1'$, $P_1$ envies the player. A similar situation occurs at the assignment of $L$. $P_a$ cuts $L$ into three pieces $L_1$, $L_2$, and $L_3$ such that $\mu_a(L_1) = \mu_a(L_2) = \mu_a(L_3)$.

1 Begin
2 $P_1$ cuts into three pieces so that the utilities of the pieces is the same for $P_1$.
3 Let $X_1, X_2, X_3$ be the pieces where $\mu_2(X_1) \geq \mu_2(X_2) \geq \mu_2(X_3)$.
4 If $\mu_2(X_1) > \mu_2(X_2)$ Then
5 $P_2$ cuts $L$ from $X_1$ so that $\mu_2(X_1') = \mu_2(X_2)$, where $X_1' = X_1 - L$.
6 $P_3$ selects the largest (for $P_3$) among $X_1', X_2$, and $X_3$.
7 If $X_1'$ remains Then
8 $P_2$ must select $X_1'$.
9 Let $(P_a, P_b)$ be $(P_3, P_2)$.
10 Else
11 $P_2$ selects $X_2$. /* the largest for $P_2$ */
12 Let $(P_a, P_b)$ be $(P_2, P_3)$.
13 $P_1$ obtains the remaining piece.
14 If $L$ is not empty Then
15 $P_a$ cuts $L$ into three pieces so that $P_a$ considers their utilities are the same.
16 $P_b, P_1$, and $P_a$ selects one piece in this order.
17 End.

**Fig. 1.** Selfridge-Conway three-player envy-free cake-cutting protocol [18].

$P_b$ selects the best piece, say $L_1$. $P_1$ then selects one of the remaining pieces, say $L_2$. In this situation, $\mu_1(L_1) > \mu_1(L) + \mu_1(L_2)$ might occur if $\mu_1(L_3)$ is a very large negative value. In this case, $\mu_1(X_1 - L) + \mu_1(L_1) > \mu_1(X_3) + \mu_1(L_2)$ holds, where $X_3$ is the piece selected by $P_1$ at Step 13. Thus, $P_1$ envies $P_b$ and envy-freeness is not satisfied. Therefore, the Selfridge-Conway protocol cannot be used for mixed manna.

Oskui's three-player envy-free chore division protocol [18], shown in Fig. 2, cannot be used for mixed manna for similar reasons. After $P_1$ cuts the manna into $X_1$, $X_2$, and $X_3$ and $X_1$ is the best piece for $P_2$ and $P_3$, $P_2$ cannot cut the manna so that $\mu_2(X_1) = \mu_2(X_2 - E) = \mu_2(X_3 - F)$ is satisfied, if $\mu_2(X_1) > 0$, $\mu_2(X_2) < 0$, and $\mu_2(X_3) < 0$. Even if $\mu_2(X_1) < 0$, $\mu_2(X_2) < 0$, $\mu_2(X_3) < 0$ and $P_2$ cuts $E$ from $X_2$ and $F$ from $X_3$ so that $\mu_2(X_1) = \mu_2(X_2 - E) = \mu_2(X_3 - F)$ is satisfied, there can be a case when $\mu_1(E) > 0$, $\mu_1(F) > 0$, and $P_1$ envies $P_3$ by the assignment at Step 11. Therefore, protocols for mixed manna must be newly considered.

A naive envy-free assignment protocol for mixed manna is shown in [19]. First, divide the manna as follows:

- $X_{123}$ such that any portion $x \subseteq X_{123}$ satisfies $\mu_i(x) \geq 0$ for every player $P_i (i = 1, 2, 3)$.
- $X_{ij} (i, j = 1, 2, 3, i < j)$ such that any portion $x \subseteq X_{ij}$ satisfies $\mu_i(x) \geq 0$, $\mu_j(x) \geq 0$, and $\mu_k(x) < 0$ for the other player $P_k$.
- $X_i (i = 1, 2, 3)$ such that any portion $x \subseteq X_i$ satisfies $\mu_i(x) \geq 0$ and $\mu_j(x) < 0$ for $j \neq i$.
- The remaining portion $X_0$ such that any portion $x \subseteq X_0$ satisfies $\mu_i(x) < 0$ for $i = 1, 2, 3$.

1 Begin
2 $P_1$ cuts into three pieces $X_1, X_2, X_3$ so that $\mu_1(X_1) = \mu_1(X_2) = \mu_1(X_3)$ is satisfied.
3 If the best piece for $P_2$ and $P_3$ differs Then
4 $P_2$ and $P_3$ select the best piece. $P_1$ obtains the remaining piece.
5 Else /* Let $X_1$ be the best piece for $P_2$ and $P_3$. */
6 $P_2$ cuts $E$ from $X_2$ and $F$ from $X_3$ so that $\mu_2(X_1) = \mu_2(X_2 - E) = \mu_2(X_3 - F)$ is satisfied.
7 If $\mu_3(X_2 - E) \leq \mu_3(X_1)$ and $\mu_3(X_3 - F) \leq \mu_3(X_1)$ Then
8 $P_1, P_3$, and $P_2$ select one piece in this order among $X_1, X_2 - E$, and $X_3 - F$.
9 Wlog $P_1$ selects $X_2 - E$. $P_3$ selects $X_1$. $P_2$ obtains $X_3 - F$.
10 $P_2$ cuts $E$ and $F$ into three pieces $E_1, E_2, E_3$ and $F_1, F_2, F_3$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F_1) = \mu_2(F_2) = \mu_2(F_3)$ are satisfied.
11 $P_3, P_1$, and $P_2$ select one piece among $E$s and $F$s in this order.
12 Else if $\mu_3(X_1) \leq \mu_3(X_2 - E)$ and $\mu_3(X_1) \leq \mu_3(X_3 - F)$ Then
13 $P_3$ cuts $E' \subseteq E$ and $F' \subseteq F$ so that $\mu_3(X_2 - E') = \mu_3(X_3 - F') = \mu_3(X_1)$ are satisfied.
14 Execute Step 8-11 by changing the roles of $P_2$ and $P_3$ and renaming $(E', F')$ to $(E, F)$.
15 Else /* $\mu_3(X_1)$ is between $\mu_3(X_2 - E)$ and $\mu_3(X_3 - F)$. */
16 Wlog $\mu_3(X_2 - E) \leq \mu_3(X_1) \leq \mu_3(X_3 - F)$ holds.
17 $P_3$ cuts $F' \subseteq F$ so that $\mu_3(X_3 - F') = \mu_3(X_1)$ is satisfied.
18 $P_1$ selects the best piece between $X_2 - E$ and $X_3 - F'$.
19 If $P_1$ selects $X_2 - E$ Then
20 $P_2$ obtains $X_1$. $P_3$ obtains $X_3 - F'$.
21 $P_3$ cuts $E$ and $F'$ into three pieces $E_1, E_2, E_3$ and $F'_1, F'_2, F'_3$ so that $\mu_3(E_1) = \mu_3(E_2) = \mu_3(E_3)$ and $\mu_3(F'_1) = \mu_3(F'_2) = \mu_3(F'_3)$ are satisfied.
22 $P_2, P_1$, and $P_3$ select one piece among $E$s and $F'$s in this order.
23 Else /* $P_1$ selects $X_3 - F'$. */
24 $P_2$ obtains $X_2 - E$. $P_3$ obtains $X_1$.
25 $P_2$ cuts $E$ and $F'$ into three pieces $E_1, E_2, E_3$ and $F'_1, F'_2, F'_3$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F'_1) = \mu_2(F'_2) = \mu_2(F'_3)$ are satisfied.
26 $P_3, P_1$, and $P_2$ select one piece among $E$s and $F'$s in this order.
27 End

**Fig. 2.** Oskui's three-player envy-free chore division protocol [18].

Then, the Selfridge-Conway protocol is executed among all players for $X_{123}$. Divide-and-choose is executed to $X_{ij}$ between $P_i$ and $P_j$. $X_i$ is given to $P_i$. Last, three-player envy-free chore division protocol [18] is executed for $X_0$. Similar procedures can be considered for any number of players. Though this procedure achieves an envy-free assignment, the procedure to initially divide the manna is complicated. The mixed manna must be divided into the above eight pieces. Note that each of the eight pieces might not be connected. For example, disconnected multiple portions might satisfy $\mu_i(x) \geq 0$ for all players, thus $X_{123}$ might consist of multiple portions. Thus, the number of cuts to obtain the above eight pieces might be more than eight. When $P_i (i = 1, 2, 3)$ needs to cut the manna $c_i$ times to divide into non-negative regions and negative regions for $P_i$, the manna needs to be cut $c_1 + c_2 + c_3$ times in the worst case. This paper considers reducing the procedure of the initial division. The protocol in [15] is not discrete since the protocol uses a moving-knife procedure. Thus this paper proposes a new discrete protocol that does not use a moving-knife procedure.

## 4 A new protocol for mixed manna

This section shows a new three-player envy-free division protocol for mixed manna in which the number of the initial division is reduced. Initially, cut the manna as follows: $X^+$ such that any portion $x \subseteq X^+$ satisfies $\mu_1(x) \geq 0$. $X^-$ such that any portion $x \subseteq X^-$ satisfies $\mu_1(x) < 0$. $X^+ (X^-)$ is the portion with non-negative (negative) utility for $P_1$. The manna must be cut $c_1$ times. Note that by a renaming of the players, $c_1$ can be selected as $\min_i c_i$. Thus, the number of cuts necessary for the initial division is reduced compared to the naive protocol in [19]. $X^+ (X^-)$ might consist of multiple disconnected pieces. In this case, the disconnected pieces are collected to make one piece. $X^+$ and $X^-$ might contain both positive and negative portions for the other players.

The assignment of $X^+$ uses the protocol in [15]. The protocol is shown in Fig. 3, in which the Selfridge-Conway protocol is modified. Initially, $P_1$ cuts $X^+$ into three pieces. If both of $P_2$ and $P_3$ think at most one piece has a non-negative utility, an envy-free assignment is easily obtained. If $P_2$ or $P_3$ thinks that at least two pieces have a non-negative utility, the Selfridge-Conway protocol can be executed because $P_1$ thinks any portion of $X^+$ has a non-negative utility.

**Theorem 1.** *[15] The assignment result of $X^+$ by the protocol in Fig. 3 is envy-free.*

*Proof.* First, consider the case when both of $P_2$ and $P_3$ consider that at most one piece among $X_1^+$, $X_2^+$, and $X_3^+$ has a non-negative utility. Consider the subcase when both of $P_2$ and $P_3$ think the same piece, say $X_1^+$, has a non-negative utility. $P_2$ and $P_3$ execute Divide-and-choose on $X_1^+$. Let $P_2$ and $P_3$ obtain $X_{12}^+$ and $X_{13}^+$, respectively. Since $X_1^+ = X_{12}^+ \cup X_{13}^+$ and any portion of $X_1^+$ has a non-negative utility for $P_1$, $\mu_1(X_{12}^+) \leq \mu_1(X_1^+) = \mu_1(X_2^+)$ and $\mu_1(X_{13}^+) \leq \mu_1(X_1^+) = \mu_1(X_2^+)$ hold. Since $P_1$ obtains $X_2^+$ and $X_3^+$, $P_1$ does not envy $P_2$ or $P_3$. $P_2$ and $P_3$ do not envy each other because of the envy-freeness of Divide-and-choose. $P_2$ does

Begin
  $P_1$ cuts into three pieces $X_1^+, X_2^+$, and $X_3^+$ so that $\mu_1(X_1^+) = \mu_1(X_2^+) = \mu_1(X_3^+)$.
  If $P_2$ and $P_3$ consider at most one piece has a non-negative utility Then
    If $P_2$ and $P_3$ consider the same piece (say, $X_1^+$) has a non-negative utility Then
      $P_2$ and $P_3$ execute Divide-and-choose on $X_1^+$.
      $P_1$ obtains $X_2^+$ and $X_3^+$.
    Else
      Each of $P_2$ and $P_3$ obtains at most one piece with a non-negative utility.
      $P_1$ obtains the remaining piece(s).
  Else
    Let $P_2$ be a player who considers two pieces have some non-negative utility.
    Rename the pieces so that $\mu_2(X_1^+) \geq \mu_2(X_2^+) \geq \mu_2(X_3^+)$.
    Execute the Selfridge-Conway protocol from Step 4 with the three pieces.
End.

**Fig. 3.** Three-player envy-free protocol for $X^+$ [15].

not envy $P_1$, since $\mu_2(X_2^+) < 0$ and $\mu_2(X_3^+) < 0$ hold. Similarly, $P_3$ does not envy $P_1$.

Next, consider the subcase when no piece has a non-negative utility for both of $P_2$ and $P_3$. In this case, $P_2$ and $P_3$ can obtain at most one piece whose utility is not negative for the player. $P_1$ obtains the remaining pieces, which have a negative utility for both of $P_2$ and $P_3$. Thus, every player does not envy the other players.

Next, consider the case when one player, say $P_2$, thinks two pieces have a non-negative utility. In this case, the Selfridge-Conway protocol can be executed. The reason is as follows. $P_2$ can cut $L$ from $X_1^+$ if $\mu_2(X_1^+) > \mu_2(X_2^+)$ since both of these utilities are non-negative. Each player can select one piece among $X_1^{'+}$, $X_2^+$, and $X_3^+$. The assignment result is envy-free, since $P_3$ selects first, there are two equal utility pieces for $P_2$, and $P_1$ can obtain one full-size piece (Note that any portion of $X^+$ has non-negative utility for $P_1$, thus $\mu_1(X_1^{'+}) \leq \mu_1(X_1^+)$ holds). An envy-free assignment of $L$ can also be realized. Even if the utility is positive or negative, $P_a$ can cut $L$ into three pieces with the same utility. $P_a$ does not envy any other players since the three pieces have the same utility. $P_b$ does not envy any other players since $P_b$ first selects a piece. $P_1$ does not envy $P_b$ since $P_b$ does not obtain 1/3 of $X^+$ (Note again $P_1$ thinks any portion of $X^+$ has a non-negative utility). $P_1$ does not envy $P_a$ since $P_1$ selects a piece before $P_a$. □

Next, $X^-$ needs to be assigned. The protocol for $X^-$ in [15] is not a discrete protocol. This paper shows a new discrete protocol. We modify the three-player envy-free chore division protocol shown in Fig. 2. The detailed protocol is shown in Fig. 4. The main differences between Oskui's protocol are these three points:

1. any portion $x \subseteq X^-$ has a negative utility for $P_1$ by the definition.
2. At Step 5, when the best piece ($X_1^-$) is the same for $P_2$ and $P_3$, both players must have a negative utility for $X_1^-$.
3. At Step 13, when $P_2$ cuts $E$ from $X_2^-$ and $F$ from $X_3^-$, $E$ and $F$ must be selected so that any portion of $E \cup F$ has a negative utility for $P_2$.

The reason for the necessity of the conditions is as follows: (1) $P_1$ must not envy for the assignment at Step 7, Step 10, Step 18, Step 29, and Step 33. The detail is shown in the proof. (2) At step 13, $P_2$ must be able to cut $E$ from $X_2^-$ and $F$ from $X_3^-$ so that $\mu_2(X_1^-) = \mu_2(X_2^- - E) = \mu_2(X_3^- - F)$, $\mu_2(X_1^-) < 0$, $\mu_2(X_2^-) < 0$, and $\mu_2(X_3^-) < 0$ are satisfied. For example, if $\mu_2(X_1^-) > 0$ and $\mu_2(X_2^-) < 0$, cutting $E$ might not be able to be executed. (3) If $E$ or $F$ has a portion whose utility is positive for $P_2$, when $P_3$ cuts $E'$ and $F'$ at Step 20, the new $X_2^- - E'$ or $X_3^- - F'$ might become the best piece for $P_2$ and an envy-free assignment cannot be obtained at Step 15.

**Theorem 2.** *The assignment result of $X^-$ by the protocol in Fig. 4 is envy-free.*

*Proof.* In the protocol, $P_1$ cuts $X^-$ into three pieces $X_1^-$, $X_2^-$, and $X_3^-$ so that $\mu_1(X_1^-) = \mu_2(X_2^-) = \mu_2(X_3^-) < 0$ is satisfied. First, $P_2$ and $P_3$ chooses the best piece. If the best pieces differ, an envy-free assignment is achieved when $P_2$ and $P_3$ selects its best piece and $P_1$ obtains the remaining piece. Note that even if $P_2$ or $P_3$ has more than one best piece, an envy-free assignment exists. For example, if $P_2$ has more than one best piece, $P_3$, $P_2$, and $P_1$ selects one piece in this order. After $P_3$ selects one piece, there is at least one remaining piece whose utility is the best for $P_2$. Thus, an envy-free assignment can be achieved.

Therefore, the remaining case to consider is $P_2$ and $P_3$ have the same best piece, say $X_1^-$. We assume that $\mu_2(X_1^-) < 0$ and $\mu_3(X_1^-) < 0$ are satisfied. Otherwise, $X_1^-$ can be assigned to the players who have a non-negative utility by the Steps 6-11. If $\mu_2(X_1^-) \geq 0$ and $\mu_3(X_1^-) \geq 0$, $X_1^-$ is divided between $P_2$ and $P_3$ using Divide-and-choose. If one of $P_2$ or $P_3$ has a non-negative utility to $X_1^-$, $X_1^-$ is given to the player without envy. Then, the procedure is executed again for the remaining pieces.

Thus, we assume that $\mu_2(X_1^-) < 0$ and $\mu_3(X_1^-) < 0$. Since $X_1^-$ is the best piece for $P_2$ and $P_3$, $\mu_2(X_2^-) \leq \mu_2(X_1^-) < 0$, $\mu_2(X_3^-) \leq \mu_2(X_1^-) < 0$, $\mu_3(X_2^-) \leq \mu_3(X_1^-) < 0$, and $\mu_3(X_3^-) \leq \mu_3(X_1^-) < 0$ are satisfied. Note that $X^-$ might have some portions whose utility is positive for $P_2$ and/or $P_3$. $P_2$ cuts $E$ from $X_2^-$ and $F$ from $X_3^-$ so that $\mu_2(X_1^-) = \mu_2(X_2^- - E) = \mu_2(X_3^- - F)$ with the condition that any portion $x \subseteq E \cup F$ satisfies $\mu_2(x) < 0$. $P_2$ can execute this operation because $\mu_2(X_2^-) \leq \mu_2(X_1^-) < 0$ and $\mu_2(X_3^-) \leq \mu_2(X_1^-) < 0$. Note that $E$ and $F$ might not be a connected component. In that case, connect these pieces and treat $E$ and $F$ as a single piece. As the original Oskui's protocol, we consider the following three cases.

(Case 1) $\mu_3(X_2^- - E) \leq \mu_3(X_1^-)$ and $\mu_3(X_3^- - F) \leq \mu_3(X_1^-)$.
In this case, $P_1$, $P_3$, and $P_2$ selects one piece in this order among $X_1^-$, $X_2^- - E$,

1 Begin
2 $P_1$ cuts $X^-$ into three pieces $X_1^-, X_2^-$, and $X_3^-$ so that $\mu_1(X_1^-) = \mu_2(X_2^-) = \mu_2(X_3^-)$ is satisfied.
3 If the best piece for $P_2$ and $P_3$ differs Then
4 $P_2$ and $P_3$ selects the best piece. $P_1$ obtains the remaining piece.
5 Else /* Let $X_1^-$ be the best piece for $P_2$ and $P_3$. */
6 If $\mu_2(X_1^-) \geq 0$ and $\mu_3(X_1^-) \geq 0$ Then
7 Execute Divide-and-choose on $X_1^-$ between $P_2$ and $P_3$.
8 Let $X^- = X_2^- \cup X_3^-$ and goto 2:
9 Else if $\mu_2(X_1^-) \geq 0$ or $\mu_3(X_1^-) \geq 0$ Then
10 Assign $X_1^-$ to the player who thinks $\mu(X_1^-) \geq 0$.
11 Let $X^- = X_2^- \cup X_3^-$ and goto 2:
12 Else /* $\mu_2(X_1^-) < 0$ and $\mu_3(X_1^-) < 0$. */
13 $P_2$ cuts $E$ from $X_2^-$ and $F$ from $X_3^-$ so that $\mu_2(X_1^-) = \mu_2(X_2^- - E) = \mu_2(X_3^- - F)$ is satisfied with the condition that any portion $x \subset E \cup F$ satisfies $\mu_2(x) < 0$.
14 If $\mu_3(X_2^- - E) \leq \mu_3(X_1^-)$ and $\mu_3(X_3^- - F) \leq \mu_3(X_1^-)$ Then
15 $P_1, P_3$, and $P_2$ select one piece in this order among $X_1^-, X_2^- - E$, and $X_3^- - F$.
16 Wlog $P_1$ selects $X_2^- - E$. $P_3$ selects $X_1^-$. $P_2$ obtains $X_3^- - F$.
17 $P_2$ cuts $E$ and $F$ into three pieces $E_1, E_2, E_3$ and $F_1, F_2, F_3$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F_1) = \mu_2(F_2) = \mu_2(F_3)$ are satisfied.
18 $P_3, P_1$, and $P_2$ select one piece among $E$s and $F$s in this order.
19 Else if $\mu_3(X_1^-) \leq \mu_3(X_2^- - E)$ and $\mu_3(X_1^-) \leq \mu_3(X_3^- - F)$ Then
20 $P_3$ cuts $E' \subseteq E$ and $F' \subseteq F$ so that $\mu_3(X_2^- - E') = \mu_3(X_3^- - F') = \mu_3(X_1^-)$ are satisfied.
21 Execute Step 15-18 by changing the roles of $P_2$ and $P_3$ and renaming $(E', F')$ to $(E, F)$.
22 Else /* $\mu_3(X_1^-)$ is between $\mu_3(X_2^- - E)$ and $\mu_3(X_3^- - F)$. */
23 Wlog $\mu_3(X_2^- - E) \leq \mu_3(X_1^-) \leq \mu_3(X_3^- - F)$ holds.
24 $P_3$ cuts $F' \subseteq F$ so that $\mu_3(X_3^- - F') = \mu_3(X_1^-)$ is satisfied.
25 $P_1$ selects the best piece between $X_2^- - E$ and $X_3^- - F'$.
26 If $P_1$ selects $X_2^- - E$ Then
27 $P_3$ obtains $X_3^- - F'$. $P_2$ obtains $X_1^-$.
28 $P_3$ cuts $E$ and $F'$ into three pieces $E_1, E_2, E_3$ and $F_1', F_2', F_3'$ so that $\mu_3(E_1) = \mu_3(E_2) = \mu_3(E_3)$ and $\mu_3(F_1') = \mu_3(F_2') = \mu_3(F_3')$ are satisfied.
29 $P_2, P_1$, and $P_3$ select one piece among $E$s and $F'$s in this order
30 Else /* $P_1$ selects $X_3^- - F'$. */
31 $P_2$ obtains $X_2^- - E$. $P_3$ obtains $X_1^-$.
32 $P_2$ cuts $E$ and $F'$ into three pieces $E_1, E_2, E_3$ and $F_1', F_2', F_3'$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F_1') = \mu_2(F_2') = \mu_2(F_3')$ are satisfied.
33 $P_3, P_1$, and $P_2$ select one piece among $E$s and $F'$s in this order
34 End.

**Fig. 4.** Three-player envy-free division protocol for $X^-$.

and $X_3^- - F$. $P_1$ never selects $X_1^-$ since every portion of $X_2^- \cup X_3^-$ has a negative utility for $P_1$. Without loss of generality, suppose that $P_1$ selects $X_2^- - E$. $P_3$ selects $X_1^-$ since it is the best piece for $P_3$. Thus $P_2$ obtains $X_3^- - F$. This assignment is envy-free. $P_1$ does not envy since $P_1$ selects first. $P_2$ does not envy since $P_2$ thinks the three pieces have the same utility.

Last, $E$ and $F$ need to be assigned. $P_2$ cuts $E$ and $F$ into three pieces $E_1$, $E_2$, $E_3$ and $F_1$, $F_2$, $F_3$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F_1) = \mu_2(F_2) = \mu_2(F_3)$ are satisfied. $P_3$, $P_1$ and $P_2$ select one piece among $E$s and $F$s in this order. $P_3$ does not envy the other players since $P_3$ selects first. $P_2$ does not envy the other players since $P_2$ thinks the utilities of three pieces are the same. $P_1$ does not envy $P_2$ since $P_1$ selects earlier than $P_2$. The reason why $P_1$ does not envy $P_3$ is as follows: $\mu_1(E) \leq \mu_1(F)$ is satisfied since $P_1$ selects $X_2^- - E$. Thus, by the selection of a piece of $E$ and $F$, $P_1$ obtains even in the worst case $1/2(\mu_1(E)+\mu_1(F)) \geq \mu_1(E)$, since every portion of $X^-$ has a negative utility for $P_1$. Thus, $P_1$ obtains in the worst case $\mu_1(X_2^- - E)+\mu_1(E) = \mu_1(X_2^-) = \mu_1(X_1^-)$. Therefore, $P_1$ does not envy $P_3$ who obtains $X_1^-$ and some pieces of $E$ and $F$. Note again the utilities of any portion of $E$ and $F$ are negative for $P_1$.

(Case 2) $\mu_3(X_1^-) \leq \mu_3(X_2^- - E)$ and $\mu_3(X_1^-) \leq \mu_3(X_3^- - F)$.
$P_3$ cuts $E' \subseteq E$ and $F' \subseteq F$ that satisfy $\mu_3(X_2^- - E') = \mu_3(X_1^-)$ and $\mu_3(X_3^- - F') = \mu_3(X_1^-)$. Such a cut is possible since $\mu_3(X_2^-) \leq \mu_3(X_1^-) < 0$ and $\mu_3(X_2^-) \leq \mu_3(X_1^-) < 0$ are satisfied. Since any portion of $E \cup F$ has a negative utility for $P_2$, $\mu_2(X_1^-) \geq \mu_2(X_2^- - E')$ and $\mu_2(X_1^-) \geq \mu_2(X_3^- - F')$ are satisfied. Now, rename $E'$ to $E$ and $F'$ to $F$. Then, the condition of (Case 1) is satisfied by $P_2$. Thus, by changing the roles of $P_2$ and $P_3$, the procedure of (Case 1) can be executed and an envy-free assignment can be obtained.

(Case 3) $\mu_3(X_1^-)$ is between $\mu_3(X_2^- - E)$ and $\mu_3(X_3^- - F)$.
Without loss of generality, suppose that $\mu_3(X_2^- - E) \leq \mu_3(X_1^-) \leq \mu_3(X_3^- - F)$ holds. In this case, $P_3$ cuts $F' \subseteq F$ that satisfies $\mu_3(X_3^- - F') = \mu_3(X_1^-)$. This operation is possible for $P_3$ since $\mu_3(X_3^-) \leq \mu_3(X_1^-) < 0$ is satisfied. $P_1$ selects the best piece between $X_2^- - E$ and $X_3^- - F'$. Note that $X_1^-$ cannot be the best piece for $P_1$ since $\mu_1(X_1^-) = \mu_1(X_2^-) = \mu_1(X_3^-)$.

(Case 3-1) $P_1$ selects $X_2^- - E$.
In this subcase, $P_3$ obtains $X_3^- - F'$. $P_2$ obtains $X_1^-$. $P_1$ does not envy the other players since $P_1$ selects first. $P_3$ does not envy the other players since $\mu_3(X_3^- - F') = \mu_3(X_1^-) \geq \mu_3(X_2^- - E)$ holds. $P_2$ does not envy the other players since $\mu_2(X_1^-) = \mu_2(X_2^- - E) \geq \mu_2(X_3^- - F')$ is satisfied.

Last, $E$ and $F'$ need to be assigned. $P_3$ cuts $E$ and $F'$ into three pieces $E_1$, $E_2$, $E_3$ and $F_1'$, $F_2'$, $F_3'$ so that $\mu_3(E_1) = \mu_3(E_2) = \mu_3(E_3)$ and $\mu_3(F_1') = \mu_3(F_2') = \mu_3(F_3')$ are satisfied. $P_2$, $P_1$, and $P_3$ select one piece among $E$s and $F'$s in this order.

Envy-freeness for $P_3$ and $P_2$ are the same as the reason of (Case 1). The reason why $P_1$ does not envy $P_2$ is as follows: $\mu_1(E) \leq \mu_1(F')$ is satisfied since $P_1$ selects $X_2^- - E$. Thus, by the selection of a piece of $E$ and $F'$, $P_1$ obtains even in the worst case $1/2(\mu_1(E) + \mu_1(F')) \geq \mu_1(E)$. Thus, $P_1$ obtains in the worst case $\mu_1(X_2^- - E) + \mu_1(E) = \mu_1(X_2^-) = \mu_1(X_1^-)$. Therefore, $P_1$ does not envy $P_2$ who obtains $X_1^-$ and some pieces of $E$ and $F'$.

(Case 3-2) $P_1$ selects $X_3^- - F'$

In this subcase, $P_2$ obtains $X_2^- - E$. $P_3$ obtains $X_1^-$. $P_1$ does not envy the other players since $P_1$ selects first. $P_3$ does not envy to the other players since $\mu_3(X_3^- - F') = \mu_3(X_1^-) \geq \mu_3(X_2^- - E)$ holds. $P_2$ does not envy the other players since $\mu_2(X_1^-) = \mu_2(X_2^- - E) \geq \mu_2(X_3^- - F')$ is satisfied.

Last, $E$ and $F'$ need to be assigned. $P_2$ cuts $E$ and $F'$ into three pieces $E_1$, $E_2$, $E_3$ and $F_1'$, $F_2'$, $F_3'$ so that $\mu_2(E_1) = \mu_2(E_2) = \mu_2(E_3)$ and $\mu_2(F_1') = \mu_2(F_2') = \mu_2(F_3')$ are satisfied. $P_3$, $P_1$ and $P_2$ select one piece among $E$s and $F'$s in this order. Envy-freeness for $P_3$ and $P_2$ are the same as the reason of (Case 1). The reason why $P_1$ does not envy $P_3$ is as follows: $\mu_1(E) \geq \mu_1(F')$ is satisfied since $P_1$ selects $X_3^- - F'$. Thus, by the selection of a piece of $E$ and $F'$, $P_1$ obtains even in the worst case $1/2(\mu_1(E) + \mu_1(F')) \geq \mu_1(F')$. Thus, $P_1$ obtains in the worst case $\mu_1(X_3^- - F') + \mu_1(F') = \mu_1(X_3^-) = \mu_1(X_1^-)$. Therefore, $P_1$ does not envy $P_3$ who obtains $X_1^-$ and some pieces of $E$ and $F'$. □

## 5 Conclusion

This paper showed a three-player discrete envy-free division protocol for mixed manna. This protocol reduces the initial division by the naive protocol. Note that the initial division still needs $\min_i c_i$ cuts, thus elimination of the initial division is the most important open problem. Also, each player's role in the protocol differs among the players and meta-envy [13] exists. A meta-envy-free protocol is necessary for ideal fairness.

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